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An Asymptotic Formula of Modified Family of Positive Linear Operators

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Abstract

In 2016, Patel and Mishra introduce the operators which is generalization of well-known Szasz-Mirakyan operators. In this manuscript, we have discussed Voronovskaja asymptotic of Stancu type generalization of the operators defined by Patel and Mishra.
Keywords—Positive linear operators; Asymptotic formula; Sazas-Mirakyan operators

1. Introduction

Using Lagrange's formula, Patel and Mishra [1] defined the following sequence of positive linear operators, for $f \in C([0,\infty)); 0 \leq \mu < 1; 1 < \gamma \leq e$ as

$$P_n^{[\mu,\gamma]}(f;x) = \sum_{k=0}^{\infty} \omega_{n,\gamma}(k;nx) f\left(\frac{k}{n}\right) \tag{1}$$

where

$$\omega_{n,\gamma}(k,nx) = nx(\log\gamma)^k (nx+k\mu)^{k-1} \frac{\gamma^{-(nx+k\mu)}}{(k!)}.$$

In particular $\gamma = e$, the operators (1) reduce to Jain operators [2]. Also, if $\gamma = e$ and $\mu = 0$ then, the operators $P_n^{[\mu,\gamma]}$ equal to the classical Szasz-Mirakyan operators [3]. Approximation properties of the Szasz-Mirakyan operators, Jain operators and their generalizations was discussed by many authors. We mention that, approximation properties of the integral generalization of Szasz-Mirakyan operators discussed in [4, 5] and integral type generalization of Jain operators discussed in [6, 7, 8]. The generalization of Szasz-Mirakyan operators based on q-integer was established in [9, 10, 11]. This research proved that the Szasz-Mirakyan operators and their generalization have many interesting approximation properties.

In 1983, the following type generalization of Bernstein polynomial was established by Stancu in [12] and studied the positive linear operators $S_n^{\alpha,\beta}: C([0,1]) \to C([0,1])$ defined for any $f \in C([0,1])$ as follows:

$$S_n^{\alpha,\beta}(f,x) = \sum_{k=0}^{n} p_{(n,k)}(x)\, f\left(\frac{k+\alpha}{n+\beta}\right), \quad 0 \le x \le 1,$$

where $p_{(n,k)}(x) = \binom{n}{k} x^k (1-x)^{n-k}$is the Bernstein basis function. After the work of Stancu many researcher work in this direction. The recent work on such type of operators can be found in [13, 14, 15, 16, 17, 18, 19, 20, 21]. This motivated us to generalize the operators (1) in the following way, for $f \in C\big([0,\infty)\big); 0 \le \mu < 1; 1 < \gamma \le e, 0 \le \alpha \le \beta$ as

$$P_n^{[\mu,\gamma,\alpha,\beta]}(f;x) = \sum_{k=0}^{\infty} \omega_{n,\gamma}(k;nx) f\left(\frac{k+\alpha}{n+\beta}\right), \tag{2}$$

where $\omega_{n,\gamma}(k;nx)$ as defined in (3). The above generalization known as Stancu type generalization of the operators (1). In particular $\alpha = \beta = 0,$ the operators (2) reduce to the operators $P_n^{[\mu,\gamma]}$.

2. Some Lemmas

To discuss moments of the operators (2), we need following lemmas:

Lemma 1([*1*]). The operators $P_n^{[\mu,\gamma]}, n > 1$, defined by (1) satisfy the following relations:

1. $P_n^{[\mu,\gamma]}(1,x) = 1;$

2. $P_n^{[\mu,\gamma]}(t,x) = \frac{x \log\gamma}{1-\mu\log\gamma};$

3. $P_n^{[\mu,\gamma]}(t^2,x) = \frac{x^2(\log\gamma)^2}{(1-\mu\log\gamma)^2} + \frac{x\log\gamma}{n(1-\mu\log\gamma)^3};$

4. $P_n^{[\mu,\gamma]}(t^3,x) = \frac{x^3(\log\gamma)^3}{(1-\mu\log\gamma)^3} + \frac{3x^2(\log\gamma)^2}{n(1-\mu\log\gamma)^4} + \frac{x\log\gamma(1+2\mu\log\gamma+\quad{}^4(\log\gamma)^3-2\mu^4(\log\gamma)^4)}{n^2(1-\mu\log\gamma)^5}$

5. $P_n^{[\mu,\gamma]}(t^4,x) = \frac{x^4(\log\gamma)^4}{(1-\mu\log\gamma)^4} + \frac{6x^3(\log\gamma)^3}{n(1-\mu\log\gamma)^5} + \frac{x^2(\log\gamma)^2\,(7+8\mu\log\gamma+\quad{}^4(\log\gamma)^3-2\mu^4(\log\gamma)^4)}{n^2(1-\mu\log\gamma)^6}$

$$+\frac{x\log\gamma\Big(1+8\mu\log\gamma+6\mu^2(\log\gamma)^2+(12\mu^4(\log\gamma)^3-16\mu^5(\log\gamma)^4+6\mu^6(\log\gamma)^5)(1-\log\gamma)\Big)}{n^3(1-\mu\log\gamma)^7}.$$

Lemma 2. The operators $P_n^{[\mu,\gamma,\alpha,\beta]}, n > 1$, defined by (1) satisfy the following relations:

1. $P_n^{[\mu,\gamma,\alpha,\beta]}(1,x) = 1;$
2. $P_n^{[\mu,\gamma,\alpha,\beta]}(\mathrm{t},x) = \frac{nx\log\gamma+\alpha(1-\mu\,\mathrm{lo}\quad)}{(n+\beta)(1-\mu\,\mathrm{lo}\quad)};$
3. $P_n^{[\mu,\gamma,\alpha,\beta]}(\mathrm{t}^2,x) = \frac{n^2x^2(\log\gamma)^2}{(n+\beta)^2(1-\mu\log\gamma)^2} - \frac{nx\log\quad(1+2\alpha)}{(n+\beta)^2(1-\mu\log\gamma)} + \frac{\alpha^2}{(n+\beta)^2}.$

Proof. It is clear that $P_n^{[\mu,\gamma,\alpha,\beta]}(1,x) = 1$. By simple computation, we get

$$P_n^{[\mu,\gamma,\alpha,\beta]}(\mathrm{t},x)=\sum_{k=0}^{\infty}\omega_{n,\gamma}(k;nx)\left(\frac{k+\alpha}{n+\beta}\right)=\frac{n}{n+\beta}P_n^{\mu,\gamma}(t,x)+\frac{\alpha}{n+\beta}=\frac{nx\log\gamma+\alpha(1-\mu\log\gamma)}{(n+\beta)(1-\mu\log\gamma)}.$$

Now, $P_n^{[\mu,\gamma,\alpha,\beta]}(\mathrm{t}^2,x)=\sum_{k=0}^{\infty}\omega_{n,\gamma}(k;nx)\left(\frac{k+\alpha}{n+\beta}\right)^2$

$$=\frac{\mathrm{n}^2}{(n+\beta)^2}P_n^{[\mu,\gamma]}(t^2,x)+\frac{2\alpha n}{(n+\beta)^2}P_n^{[\mu,\gamma]}(t,x)+\frac{\alpha^2}{(n+\beta)^2}$$

$$=\frac{n^2x^2(\log\gamma)^2}{(n+\beta)^2(1-\mu\log\gamma)^2}-\frac{nx\log\gamma\,(1+2\alpha)}{(n+\beta)^2(1-\mu\log\gamma)}+\frac{\alpha^2}{(n+\beta)^2},$$

we have the desired result.

Remark 1. For all $m\in\mathbb{N},0\le\alpha\le\beta$; we have the following recursive relation for the images of the monomials t^m under $P_n^{[\mu,\gamma,\alpha,\beta]}(t^m,x)$ in terms of $P_n^{[\mu,\gamma]}(t^j,x),\ j=0,1,2,\dots,m$ as

$$P_n^{[\mu,\gamma,\alpha,\beta]}(t^m,x)=\sum_{j=0}^{m}\binom{m}{j}\frac{n^j\alpha^{m-j}}{(n+\beta)^m}P_n^{[\mu,\gamma]}(t^j,x).$$

Remark 2. We have

$$\Phi_n^{[\mu,\gamma,\alpha,\beta]}(x)=P_n^{[\mu,\gamma,\alpha,\beta]}(t-x,x)=x\left(\frac{n(\log\gamma-1+\mu\log\gamma)-\beta(1-\mu\log\gamma)}{(n+\beta)(1-\mu\log\gamma)}\right)+\frac{\alpha}{(n+\beta)};$$

$$\begin{aligned}\Psi_n^{[\mu,\gamma,\alpha,\beta]}(x)&=P_n^{[\mu,\gamma,\alpha,\beta]}((t-x)^2,x)\\&=x^2\left(\frac{\left(\beta(1-\mu\log\gamma)+n(1-\log\gamma-\mu\log\gamma)\right)^2}{(n+\beta)^2(1-\mu\log\gamma)^2}\right)\\&+x\left(\frac{\left(n\left((1+2\alpha+2\alpha\mu)\log\gamma-2\alpha\right)\right)}{(n+\beta)^2(1-\mu\log\gamma)}\right)\\&+x\left(\frac{-2\alpha\beta(1-\mu\log\gamma)}{(n+\beta)^2(1-\mu\log\gamma)}\right)+\frac{\alpha^2}{(n+\beta)^2}.\end{aligned}$$

3. Voronovskaja Type Theorem

In this section, we establish the asymptotic formula for the operators $P_n^{[\mu,\gamma,\alpha,\beta]}$.

Theorem 1. For $b>0,\mu_n\in(0,1)$ such that $n\mu_n\to l\in\mathbb{R}$ and $\gamma_n\in(1,e)$ such that $\gamma_n\to e$(Euler number). Then for every $f\in C([0,b]),f',f''$ exists at a fixed point $x\in(0,b)$, we have

$$\lim_{n\to\infty}n\left(P_n^{[\mu_n,\gamma_n,\alpha,\beta]}(f,x)-f(x)\right)=\ (\alpha+(l-\beta)x)f'(x)+\frac{((l^2+2\beta)x+1)x}{2}f''(x).$$

Proof. Let $x\in(0,b)$ be fixed. From the Taylor's theorem, we may write

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+r(t,x)(t-x)^2, \tag{4}$$

where $r(t,x)$ is the peano form of the remainder and $\lim_{\mathrm{t}\to\mathrm{x}}r(t,x)=0.$

Applying $P_n^{[\mu,\gamma,\alpha,\beta]}$on the both side of equation (4), we have

$$n\left(P_n^{[\mu,\gamma,\alpha,\beta]}(f,x)-f(x)\right)=nf'(x)\Phi_n^{[\mu,\gamma,\alpha,\beta]}(x)+\frac{1}{2}nf''(x)\Psi_n^{[\mu,\gamma,\alpha,\beta]}(x).$$

In view of Remark 1, we have

$$\lim_{n\to\infty} n\Phi_n^{[\mu,\gamma,\alpha,\beta]}(x)=\alpha+(l-\beta)x; \tag{5}$$

$$\lim_{n\to\infty} n\Psi_n^{[\mu,\gamma,\alpha,\beta]}(x)=((l^2+2\beta)x+1)x. \tag{6}$$

Now, we shall show that

$$\lim_{n\to\infty} \mathrm{n}\, P_n^{[\mu,\gamma,\alpha,\beta]}(r(t,x)(t-x)^2,x) \ =0.$$

By using Cauchy-Schwarz inequality, we have

$$P_n^{[\mu,\gamma,\alpha,\beta]}(r(t,x)(t-x)^2,x)\le\left(P_n^{[\mu,\gamma,\alpha,\beta]}(r^2(t,x),x)\right)^{\frac{1}{2}}\left(P_n^{[\mu,\gamma,\alpha,\beta]}((t-x)^4,x)\right)^{\frac{1}{2}}. \tag{7}$$

We observe that $r^2(x,x)=0$ and $r^2(\cdot,x)\in C([0,b])$. Then, it follows that

$$\lim_{n\to\infty} P_n^{[\mu,\gamma,\alpha,\beta]}(r^2(t,x),x)=r^2(x,x)=0, \tag{8}$$

in view of the fact that $P_n^{[\mu,\gamma,\alpha,\beta]}((t-x)^4,x)=O\left(\frac{1}{n^2}\right)$.

Now, from (7) and (8), we obtain

$$\lim_{n\to\infty} nP_n^{[\mu,\gamma,\alpha,\beta]}(r(t,x)(t-x)^2,x)=0. \tag{9}$$

From (5), (6) and (9), we get the required result.

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References

[1] P Patel and V. N. Mishra, "On Generalized Szasz-Mirakyan operators," *arXiv preprint arXiv:1508.07896,* 2015.

[2] G. C. Jain, "Approximation of functions by a new class of linear operators," *J. Aust. Math. Soc.,* vol. 13, no. 3, pp. 271-276, 1972

[3] O. Szasz, "Generalization of S. Bernsteins polynomials to the infinite interval," *J. Res. Natl. Bur. Stand.,* vol. 45, pp. 239-245, 1950.

[4] O. Duman, M. A. Ozarslan and H. Aktuglu, "Better error Esimation for Szasz-Mirakjan-Beta Operators," *Journal of Computational Analysis & Applications,* vol. 10, no. 1, 2008.

[5] H.M. Srivastava and V Gupta , "A certain family of summation-integral type operators," *Mathematical and Computer Modelling,* vol. 37, no. 12, pp. 1307-1315, 2003.

[6] S. Tarabie, "On Jain-beta linear operators," *Appl. Math. Inf. Sci,* vol. 6, no. 2, pp. 213-216, 2012.

[7] P. Patel and V N Mishra, "Jain-Baskakov operators and its different generalization," *Acta Mathematica Vietnamica,* vol. 4, no. 715-733, p. 40, 2015.

[8] V N Mishra and and P. Patel, "Some approximation properties of modified Jain-Beta operators," *Journal of Calculus of Variations,* vol. 2013, 2013.

[9] N. I. Mahmudov, "On q-parametric Szasz-Mirakjan operators," *Mediterranean journal of mathematics,* vol. 7, no. 3, pp. 297-211, 2010.

[10] N. I. Mahmudov, "Approximation properties of complex q-Szasz--Mirakjan operators in compact disks," *Computers & Mathematics with Applications,* vol. 60, no. 6, pp. 1784-1791, 2010.

[11] A. Aral, "A generalization of Szasz--Mirakyan operators based on q-integers," *Mathematical and Computer Modelling,* vol. 47, no. 9, pp. 1052-1062, 2008.

[12] D D Stancu, "Approximation of functions by means of a new generalized," *Calcolo,* vol. 20, pp. 211-229, 1983.

[13] Büyükyazıcı, İ. and Atakut, Ç, "On Stancu type generalization of q-Baskakov operators," *Mathematical and Computer Modelling,* vol. 52, no. 5, pp. 752-759, 2010.

[14] Ç Atakut, İ Büyükyazıcı, "Stancu type generalization of the Favard--Szasz operators," *Applied Mathematics Letters,* vol. 23, no. 12, pp. 1479-1482, 2010.

[15] V. N. Mishra and P. Patel, "Approximation properties of q-Baskakov-Durrmeyer-Stancu operators," *Mathematical Sciences,* vol. 7, no. 1, pp. 1-12, 2013.

[16] O. Dougru. R M Mohapatra and M Örkcu, "Approximation Properties of Generalized Jain Operators," *Filomat,* vol. 30, no. 9, pp. 2359-2366, 2016

[17] V. N. Mishra and P. Patel, "The Durrmeyer type modification of the q-Baskakov type operators with two parameter α and β," *Numerical Algorithms,* vol. 67, no. 4, pp. 753-769, 2014.

[18] V. N. Mishra and P. Patel, "A short note on approximation properties of Stancu generalization of q-Durrmeyer operators," *Fixed point theory and Applications,* vol. 2013, no. 1, pp. 1--5, 2013.

[19] V. N.Mishra, K. Khatri and L. N. Mishra, Deepmala, "Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators", Journal of Inequalities and Application, vol. 2013, no. 1, pp. 586, 2013.

[20] V. N. Mishra, K. Khatri and L. N. Mishra, On Simultaneous Approximation for Baskakov-Durrmeyer-Stancu type operators, Journal of Ultra Scientist of Physical Sciences, Vol. 24, No. (3) A, pp. 567-577, 2012.

[21] V. N. Mishra, H. H. Khan, K. Khatri and L. N. Mishra, Hypergeometric Representation for Baskakov-Durrmeyer-Stancu Type Operators, Bulletin of Mathematical Analysis and Applications, ISSN: 1821-129, Vol. 5, Issue 3, 18-26, 2013.