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Guaranteed phase synchronization of hybrid oscillators using symbolic Euler's method (verification challenge)

Jawher Jerray[1], Laurent Fribourg[2], and Étienne André[3]

[1] Université Sorbonne Paris Nord, LIPN, CNRS, UMR 7030, F-93430, Villetaneuse, France
[2] Université Paris-Saclay, LSV, CNRS, ENS Paris-Saclay
[3] Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France
jerray@lipn.univ-paris13.fr

Abstract

The phenomenon of *phase synchronization* was evidenced in the 17th century by Huygens while observing two pendulums of clocks leaning against the same wall. This phenomenon has more recently appeared as a widespread phenomenon in nature, and turns out to have multiple industrial applications. The exact parameter values of the system for which the phenomenon manifests itself are however delicate to obtain in general, and it is interesting to find formal sufficient conditions to *guarantee* phase synchronization. Using the notion of *reachability*, we give here such a formal method. More precisely, our method selects a portion S of the state space, and shows that any solution starting at S *returns* to S within a fixed number of periods k. Besides, our method shows that the components of the solution are then (almost) in phase. We explain how the method applies on the Brusselator reaction-diffusion and the biped walker examples. These examples can also be seen as "challenges" for the verification of continuous and hybrid systems.

1 Introduction

The phenomenon of phase synchronization was evidenced in the 17th century by Huygens while observing two pendulums of clocks leaning against the same wall. This phenomenon has more recently appeared as a widespread phenomenon in nature, and turns out to have multiple industrial applications [Win80; MS90; KZH02; Ace+05].

Basically, we consider a system consisting of two periodic coupled oscillators. After a certain time, the same period T for both oscillators is found, and, whatever the initial condition of each oscillator, the two components evolve in phase on their respective orbits.

The exact parameter values of the system for which the phenomenon manifests itself are however delicate to obtain in general, and it is interesting to find formal sufficient conditions to guarantee phase synchronization. There is a classical method, called "direct", which is used to characterize such conditions [Win80]. Basically, this method starts from a pair of synchronized components evolving on their respective orbits, then moves "slightly" apart each component (with the help of a small perturbation), and observes, after a fixed number of periods, say k, that

the phases of the two components have become very close to each other again (see e. g., [SKN17, Appendix H] for a formal description). Such a method shows besides that the synchronization is *robust* (or "stable") since, after a small disturbance, the system resynchronizes quickly (see, e. g., [Mag79]).

We will reproduce the spirit of this method using the notion of *reachability*. More precisely, our method selects a portion S of the state space, and shows that any solution starting at S *returns* to S within a fixed number of periods k. Besides, our method shows that the components of the solution are then (almost) in phase.

After a formal description of the method, we explain how the method applies on the Brusselator reaction-diffusion and the biped walker examples. These examples can also be seen as "challenges" for the verification of continuous and hybrid systems.

Plan In Section 2, we explain the underling principle of our method, which is based on the notion of reachability. We describe in Section 3 how this principle is implemented using symbolic Euler's method. We illustrate the method on the Brusselator reaction-diffusion example (Section 4) and the biped walker example (Section 5). We conclude in Section 6.

2 Showing synchronization using a reachability method

We consider a system composed of n subsystems governed by a system of differential equations (ODEs) of the form $\dot{x}(t) = f(x(t))$. For the sake of simplicity, we suppose here $n = 2$.[1] The system of ODEs is thus of the form:

$$\dot{x_1}(t) = f_1(x_1(t), x_2(t))$$
$$\dot{x_2}(t) = f_2(x_1(t), x_2(t))$$

with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^m \times \mathbb{R}^m$, where m is the dimension of the state space of each subsystem. The initial condition is of the form $(x_1^0, x_2^0) \in \mathbb{R}^m \times \mathbb{R}^m$.

The set $S = S_1 \times S_2$ (with $S_i \subset \mathbb{R}^m$, $i = 1, 2$) on which we focus our analysis, is selected roughly speaking as follows. We first consider, for each subsystem i ($i = 1, 2$), a "ring" of reduced width e_i around the cyclic trajectory (orbit). We then select a fragment of each ring, which gives two sets of states S_1 and S_2. Typically, for $i = 1, 2$, S_i is a *parallelogram* with a *horizontal* "base" of width e_i (or symmetrically a vertical side). The set S_i is thus characterized by a triple (a_i, b_i, e_i) where a_i and b_i are the end points of its main diagonal, and e_i the size of its horizontal base.[2] We assume that the parallelogram S_i is "long", i.e.:

(H) The width e_i of S_i is "small" w.r.t. $f_i = |ord(b_i) - ord(a_i)|$.

where $ord(a_i)$ (resp. $ord(b_i)$) denotes the ordinate of a_i (resp. b_i).

Typically, we have: $e_i/f_i < 1/20 = 0.05$. We now consider a point $x^0 = (x_1^0, x_2^0) \in S$ (i. e., $x_1^0 \in S_1$ and $x_2^0 \in S_2$), and consider the following procedure $PROC0(x^0)$:

1. Show that, if $x(0) = x^0$, then there exists $t \in [kT, (k+1)T)$: $x(t) \in S$ (i. e., $(x_1(t), x_2(t)) \in S_1 \times S_2$) (*recurrence* of S), and

[1] The extension of the method to $n \geqslant 3$ is straightforward in principle, but is a source of combinatorial explosion.

[2] The precise finding of the coordinates of a_i and b_i, and size e_i ($i = 1, 2$) for which our method of synchronization applies successfully, is actually a basic difficulty of the method, but this issue is beyond the scope of this paper. We assume here that a_i, b_i and e_i are given.

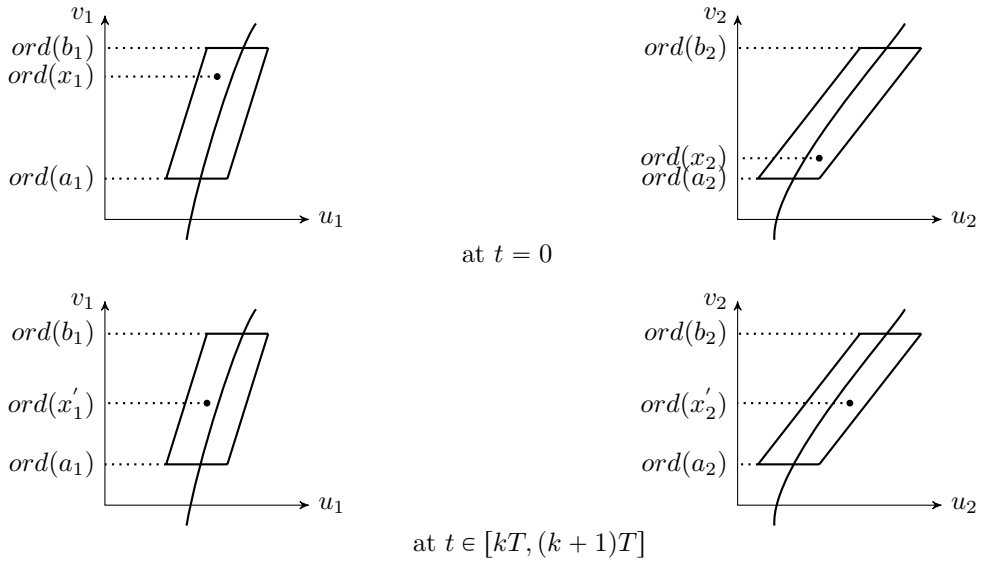

Figure 1: Scheme of S_1 (left) and S_2 (right) at $t = 0$ (top) and for some $t \in [kT, (k+1)T)$ (bottom).

2. At t, the two components $x_1(t)$ and $x_2(t)$ of $x(t)$ are practically in phase, i.e.: $|\phi(x_1(t)) - \phi(x_2(t))| < \epsilon$ (*synchronization*)

Remark 1. IN $PROC0$, we assume that T, k, ϵ are given constants, where T is the period and k is the number of periods.

Remark 2. The procedure guarantees only a *recurrent* form of synchronization at times $t, t', \ldots, t^{(n)}, \ldots$ with $nkT \leqslant t^{(n)} < n(k+1)T$. This is weaker than *standard* synchronization which states that, after the end of the perturbation, the state $x(t)$ *converges* to a solution whose components are in phase.

The notion of *phase* $\phi(x_i(s))$, for $i = 1, 2$ of component $x_i(s)$ at time s, remains to be defined in this framework. From a general point of view, one can suppose that, during its traversal of S_i, the phase of the point $x_i(s)$ varies, after normalization, between 0 and 1. As S_i is of small dimension with respect to the orbit of the subsystem i, we can assimilate the trajectory described by $x_i(s)$ in S_i to a straight line segment whose ordinate varies from $ord(a_i)$ to $ord(b_i)$. Moreover, we can assume that on this small fragment of orbit, the *phase velocity* is *constant*. Given a point of $x_i(s)$ of $S_i \equiv (a_i, b_i, e_i)$ at time s $(i = 1, 2)$, we can thus define its *phase* $\phi[x_i(s)]$ (in a "linearized" and "normalized" manner w.r.t. S_i) by:

$$\phi[x_i(s)] = (ord(x_i(s)) - ord(a_i))/(ord(b_i) - ord(a_i)),$$

See Fig. 1.

3 Symbolic reachability using Euler's method

The above procedure $PROC0$ takes a *point* of S as input. So it is not possible to prove the synchronization of *all* the points starting at S, since they are in infinite number. We thus need to consider a *symbolic* (or "set-based") version of $PROC0$ which takes a *dense subset of points* as input. Such subsets are considered here under the form of "(double) ball" of the form $B = B_1 \times B_2$, where $B_i \subset \mathbb{R}^m$ $(i = 1, 2)$ is a ball of the form $\mathcal{B}(c_i, r)$ with $c_i \in \mathbb{R}^m$ (*centre*) and r a positive real (*radius*).[3]

Let $B^0 = \mathcal{B}(c_1^0, r^0) \times \mathcal{B}(c_2^0, r^0) \subset \mathbb{R}^m \times \mathbb{R}^m$, with $c_i^0 \in \mathbb{R}^m$ $(i = 1, 2)$ and r^0 positive real. As a symbolic method, we use here the *symbolic Euler's method* [Le +17; Fri17] in order to compute (an overapproximation of) the set of solutions starting at B^0. We define for $t \geqslant 0$:

$$B^{euler}(t) = \mathcal{B}(c_1(t), r(t)) \times \mathcal{B}(c_2(t), r(t)),$$

where $(c_1(t), c_2(t)) \in \mathbb{R}^m \times \mathbb{R}^m$ is the approximated value of solution $x(t)$ of $\dot{x} = f(x)$ with initial condition $x(0) = (c_1^0, c_2^0)$ given by *Euler's explicit method*, and $r(t) \approx r^0 e^{\lambda t}$ is the *expanded* radius using the *one-sided Lipschitz constant* λ (also called "logarithmic norm" or "matrix norm") [Söd06; AS12]) associated to f (see [Fri17] for details).[4] It is shown in [Le +17] that $B^{euler}(t)$ contains all the solutions $x(t)$ that start at B^0:

$$B^{euler}(t) \supseteq \{x(t) \mid x(0) \in B^0\} \equiv \{(x_1(t), x_2(t)) \mid (x_1(0), x_2(0)) \in \mathcal{B}(c_1^0, r^0) \times \mathcal{B}(c_2^0, r^0)\}. \ (*)$$

Given a ball $B = B_1 \times B_2 \subset \mathbb{R}^m \times \mathbb{R}^m$, the symbolic version of $PROC0$ is defined as follows:

$PROC1(B)$
Let $B^0 := B$. Show that there exists $t \in [kT, (k+1)T)$:
1'. $B^{euler}(t) \subset S$, i.e.: $\mathcal{B}(c_i(t), r(t)) \subset S_i$ for $i = 1, 2$. (*recurrence*)
2'. $|phase(c_1(t)) - phase(c_2(t))| \leqslant \epsilon$ (*synchronization*)

Note that, since $\mathcal{B}(c_i(t), r(t)) \subset S_i$ $(i = 1, 2)$ by (1'), we have:

$$r(t) \leqslant \frac{1}{2} \min(e_1, e_2) \quad (**)$$

where e_i denotes the width of S_i.

Remark 3. Works by Aminzare, Sontag, Arcak and others make use of logarithmic norms to prove phase synchronization but only in a *contractive* context ($\lambda < 0$) [Arc11; AS14; Sha+13]. On the other hand, logarithmic norms (with possibly $\lambda > 0$) have been used to the symbolic control of hybrid systems [RR19; RR17; Fan+17], but not to phase synchronization.

Given S_i $(i = 1, 2)$ defined as a parallelogram (a_i, b_i, e_i), in order to show the phenomenon of phase synchronization, we first *cover* S_i with a *finite* set $\{B_{j,i}\}_{j \in J_i}$ of balls $B_{j,i} \subset \mathbb{R}^m$ (i. e., for $i = 1, 2$, $S_i \subset \bigcup_{j \in J_i} B_{j,i}$). From 1', 2', (*) and (**), it follows:

[3] $x_i \in \mathcal{B}(c_i, r)$ means $\|x_i - c_i\| \leqslant r$ where $\|\cdot\|$ is the Euclidean norm.

[4] The value of λ is defined "locally", and varies according to the position of $x(t) = (x_1(t), x_2(t))$ in the state space. For regions where $\lambda < 0$, the value of $r(t)$ is considered to be constant; the value of $r(t)$ increases only when $x(t)$ occupies a region where $\lambda > 0$ (which corresponds in Fig. 2 in case $x_1(t)$ or $x_2(t)$ is located in the *red* part of its orbit). See [Fri17].

Proposition 1. *Given a covering* $\{B_j\}_{j\in J_i}$ *of* S_i $(i = 1, 2)$*, if, for all* $(j_1, j_2) \in J_1 \times J_2$*,* $PROC1(B_{j_1} \times B_{j_2})$ *succeeds, then, for all initial condition* $(x_1^0, x_2^0) \in S$*, there exists* $t \in [kT, (k+1)T)$ *such that* $(x_1(t), x_2(t)) \in S$*. Besides:*

$|phase(x_1(t)) - phase(x_2(t))| \leqslant \epsilon + \min(e_1/f_1, e_2/f_2)$,

where e_i *is the width of* S_i*, and* $f_i = |ord(b_i) - ord(a_i)|$ *its height* $(i = 1, 2)$*.*

When $\epsilon \ll \min(e_1/f_1, e_2/f_2)$, the final difference of phase between $x_1(t)$ and $x_2(t)$ is practically upper bounded by $\min(e_1/f_1, e_2/f_2)$. Since, by (H), e_i is "small" w.r.t. f_i, we know by Proposition 1 that, if $PROC1$ succeeds for a set of balls covering S, then:

For any initial point $(x_1^0, x_2^0) \in S$, there exists $t \in [kT, (k+1)T)$ such that $x_1(t)$ and $x_2(t)$ are *almost in phase*. In particular, even if $|phase(x_1^0) - phase(x_2^0)| \approx 1$ (when x_1^0 is located near a_1 and x_2^0 near b_2, or symmetrically), we have: $|phase(x_1(t)) - phase(x_2(t))| \approx 0$.

4 Example: Brusselator Reaction-Diffusion

We consider the 1D Brusselator partial differential equation (PDE), as given in [CP93]. Here we consider a state of the form $x(y,t) = (u(y,t), v(y,t))$ where $y \in \Omega = [0, \ell]$ is the spatial location. The PDE is of the form

$$\begin{cases} \frac{\partial u}{\partial t} = A + u^2 v - (B+1)u + \sigma \nabla^2 u \\ \frac{\partial v}{\partial t} = Bu - u^2 v + \sigma \nabla^2 v \end{cases} \tag{1}$$

with boundary condition: $u(0,t) = u(\ell,t) = 1$, $v(0,t) = v(\ell,t) = 3$,
and initial condition: $x_0(y) = (u(y,0), v(y,0))$ with $u(y,0) = 1 + sin(2\pi y)$, $v(y,0) = 3$.
Let: $A = 1, B = 3, \sigma = 1/40$, $\ell = 1$. We transform the PDE into a system of ODEs by spatial discretization using a grid of $N+1$ points with $N = 4$ (i.e.: $y_i = \frac{i\ell}{N+1} = 0.2i$ for $i = 1, 2, 3, 4$). We thus consider that we have 4 oscillators of state $x(y_i, t) = (u(y_i, t), v(y_i, t))$ with initial conditions $x(y_i, 0) = (u(y_i, 0), v(y_i, 0))$ $(i = 1, 2, 3, 4)$. These oscillators are coupled by a Laplacian matrix accounting for the continuous diffusion process;the size of the resulting global ODE is $N \times n = 4 \times 2 = 8$. The system of ordinary differential equations for this example is described by

$$\begin{cases} \dot{u_1} = A + u_1^2 v_1 - (B+1)u_1 + \sigma(u_0 - 2u_1 + u_2) \\ \dot{v_1} = Bu_1 - u_1^2 v_1 + \sigma(v_0 - 2v_1 + v_2) \\ \dot{u_2} = A + u_2^2 v_2 - (B+1)u_2 + \sigma(u_1 - 2u_2 + u_3) \\ \dot{v_2} = Bu_2 - u_2^2 v_2 + \sigma(v_1 - 2v_2 + v_3) \\ \dot{u_3} = A + u_3^2 v_3 - (B+1)u_3 + \sigma(u_2 - 2u_3 + u_4) \\ \dot{v_3} = Bu_3 - u_3^2 v_3 + \sigma(v_2 - 2v_3 + v_4) \\ \dot{u_4} = A + u_4^2 v_4 - (B+1)u_4 + \sigma(u_3 - 2u_4 + u_5) \\ \dot{v_4} = Bu_4 - u_4^2 v_4 + \sigma(v_3 - 2v_4 + v_5) \end{cases} \tag{2}$$

with $u_0 = u_5 = 1$ and $v_0 = v_5 = 3$. By using symmetry, we can reduce the problem to plans $x = 0.2$ and $x = 0.4$ ($x = 0.6$ coincides with $x = 0.4$, and $x = 0.8$ with $x = 0.2$). We give in Fig. 2 a typical cyclic trajectory in plans $x = 0.2$ and $x = 0.4$, during one period T. The coordinates of the parallelepiped vertices are for plan $x = 0.2$:

$((0.621884, 3.778615), (0.621888, 3.778615), (0.621906, 3.778650), (0.621903, 3.778650))$,
and for plan $x = 0.4$:

$((0.485926, 4.077926), (0.485929, 4.077926), (0.485946, 4.077997), (0.485943, 4.077997))$.
These parallepipeds are depicted in Fig. 3 (and also at magnified scale in Fig. 2). The time-

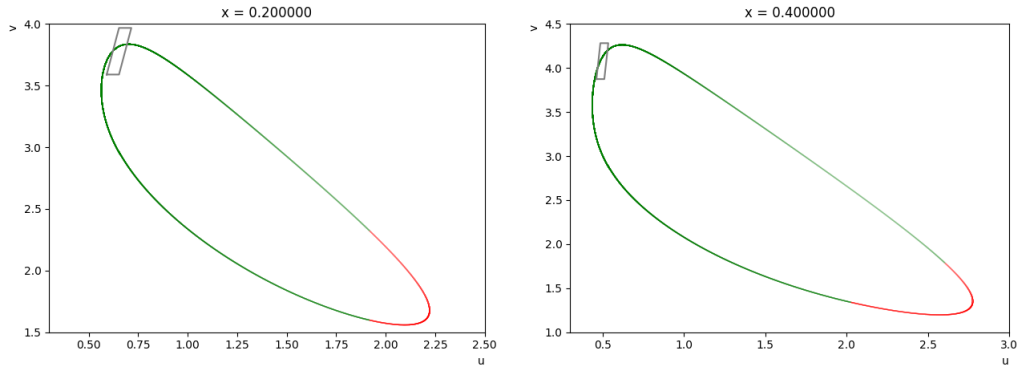

Figure 2: Brusselator: A cyclic trajectory for plan $x = 0.2$ (left) and $x = 0.4$ (right); the green zone indicates the contractive area ($\lambda < 0$) and the red zone the expansive one ($\lambda > 0$)

step used in Euler's method is $\tau = 2 \cdot 10^{-4}$, and the period of the system is $T = 34564\tau$. The expansion factor of the ball radius after one period is $E = 2.12$. The number of periods considered for synchronization is $k = 5$ (so the expansion factor after k periods $= 2.12^5 \approx 43$), and $\epsilon = 0.1$. The radius of the balls covering S is $= 3.5 \cdot 10^{-8}$.

In Fig. 3, we have depicted an initial ball (yellow) with a center of coordinate $(0.622, 3.779)$ in plan $x = 0.2$, and $(0.486, 4.078)$ in plan $x = 0.4$; its radius is $3.5 \cdot 10^{-8}$. After $k = 5$ periods, the image of the yellow ball is the green ball of center $(0.62190185, 3.77864437)$ in plan $x = 0.2$, and $(0.48594267, 4.07798666)$ in plan $x = 0.4$; the radius is now $1.5 \cdot 10^{-6}$. The phase of the initial ball center is 0.82 in plan $x = 0.2$, and 0.09 in plan $x = 0.4$, so the difference of phase $\Delta(phase(centers))$, at $t = 0$, is 0.73. The phase of the image ball center is 0.87461 in plan $x = 0.2$, and 0.87463 in plan $x = 0.4$, so the difference of phase $\Delta(phase(centers))$, after $k = 5$ periods, is now $2 \cdot 10^{-5} \approx 0$.

Fig. 4 depicts 10 (pairs of) initial balls with centers located on the parallelepiped *perimeters*, both in plan $x = 0.2$ and $x = 0.4$. The coordinates of the 10 (pairs of) centers, given under the form (u_1, v_1, u_2, v_2), are:

$((0.621890, 3.778619, 0.485930, 4.077929), (0.621895, 3.778628, 0.485928, 4.077933),$
$(0.621889, 3.778623, 0.485933, 4.077953), (0.621902, 3.778640, 0.485934, 4.077946),$
$(0.621892, 3.778629, 0.485939, 4.077966), (0.621886, 3.778620, 0.485936, 4.077966),$
$(0.621895, 3.778630, 0.485942, 4.077978), (0.621900, 3.778640, 0.485945, 4.077991),$
$(0.621905, 3.778650, 0.485939, 4.077978), (0.621902, 3.778640, 0.485942, 4.077990))$

After $k = 5$ periods, the coordinates of (u_1, v_1, u_2, v_2) become (u'_1, v'_1, u'_2, v'_2) as follows:

$((0.621897, 3.778636, 0.485938, 4.077970), (0.621899, 3.778639, 0.485940, 4.077976),$
$(0.621901, 3.778643, 0.485942, 4.077984), (0.621886, 3.778617, 0.485928, 4.077930),$
$(0.621886, 3.778617, 0.485928, 4.077929), (0.621902, 3.778645, 0.485943, 4.077988),$
$(0.621889, 3.778623, 0.485931, 4.077941), (0.621893, 3.778629, 0.485934, 4.077954),$
$(0.621892, 3.778627, 0.485933, 4.077950), (0.621893, 3.778629, 0.485934, 4.077953))$

The two components (u_1, v_1) and (u_2, v_2) of an initial point, as well as the two components (u'_1, v'_1) and (u'_2, v'_2) of its image, are all the 4 represented with the same color in Fig. 4. The CPU time taken for computing these 10 images is 4,600 seconds (for a program[5] of $PROC1$ in Python running on a 2.80 GHz Intel Core i7-4810MQ CPU with 8 GB of memory.). Table 1 gives the phases of the 10 ball centers shown in Fig. 4. After $k = 5$ periods, we have $\Delta(phase(centers)) \ll \min(e_1/f_1, e_2/f_2)$, so the difference of phase between the components of a point starting from

[5]Source codes and figures available at www.lipn.univ-paris13.fr/~jerray/synchro

Table 1: The list of phases of 10 ball centers for the Brusselator example.

Phases						
Point	phase initial point in u_1	phase initial point in u_2	phase image point in u_1	phase image point in u_2	Δ(phase (centers)) for initial point	Δ(phase (centers)) for image point
1	0.13	0.05	0.63224	0.63221	0.08	$2 \cdot 10^-5$
2	0.40	0.10	0.72512	0.72511	0.30	$8 \cdot 10^-6$
3	0.26	0.39	0.83112	0.83113	0.13	$6 \cdot 10^-6$
4	0.95	0.28	0.0383	0.0382	0.67	$9 \cdot 10^-5$
5	0.42	0.57	0.0366	0.0365	0.15	$9 \cdot 10^-5$
6	0.10	0.56	0.88834	0.88836	0.46	$1 \cdot 10^-5$
7	0.58	0.74	0.2103	0.2102	0.16	$7 \cdot 10^-5$
8	0.66	0.92	0.3929	0.3928	0.25	$5 \cdot 10^-5$
9	0.93	0.74	0.3318	0.3317	0.19	$6 \cdot 10^-5$
10	0.77	0.91	0.3890	0.3889	0.14	$5 \cdot 10^-5$

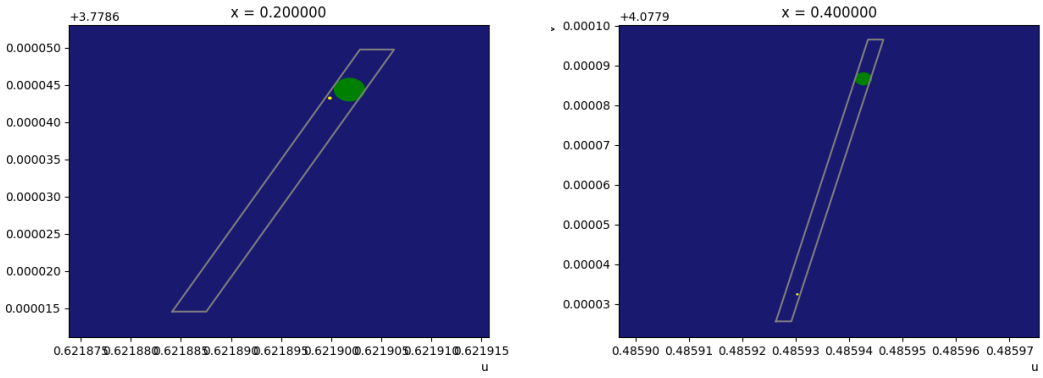

Figure 3: Brusselator: Synchronization of the two components of a ball, located initially near opposite vertices of the parallelograms (yellow), after $k = 5$ periods (green).

anywhere in a ball (not necessarily from its center) becomes always $\leqslant \min(e_1/f_1, e_2/f_2) \approx 0.05$. The proof has been done here for 10 balls, but should be done for the *whole set* of balls covering S. It is easy to see that the number of balls covering S is approximatively $\ell_1\ell_2E^{4k}/e_1e_2$, where ℓ_i is the length of each parallepiped ($i = 1, 2$). For example, if $\ell_1/e_1 = \ell_2/e_2 = 20$, $E^k = 40$, roughly as in Brusselator, the number of balls is $400 \times 40^4 = 2^{10} \cdot 10^6 \approx 10^9$, which is huge. However the analysis can be *decomposed* into k periods, and accessibility per period proven *separately* from one intermediate area to the next, thus exponentially decreasing the number of balls. In this case, the procedure has to be performed successively k times, but the number of balls at each time is now just $\ell_1\ell_2E^4/e_1e_2$, which is $400 \times 2^4 = 6400$.

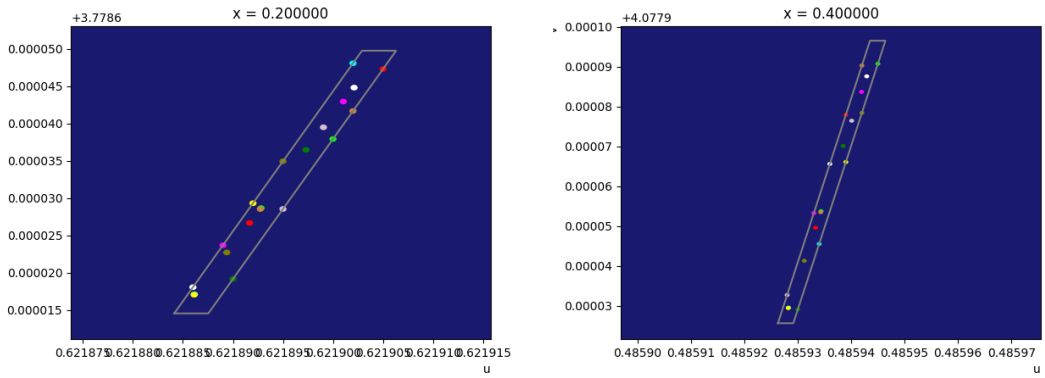

Figure 4: Brusselator: Synchronization of 10 (pairs of) balls, located initially on the parallelogram perimeters, after $k = 5$ periods (without radius expansion for clarity).

5 Example: Passive biped model

So far, we he have considered only *continuous systems* governed by ODEs. It is possible to extend the method of verification of phase synchronization to *hybrid systems*, i. e., continuous systems which, upon the satisfaction of a certain *state condition* ("guard"), may *reset* instantaneously the state before resuming the application of ODEs. Many works in the domain of symbolic control have explained how to compute an overapproximation of the *intersection* of the current set of reachability with the guard condition, and perform the reset operation (see, e. g., [GG08; AK12; KA20]). Our symbolic Euler's method can be extended along these lines without major problems. We describe here the results of such an extension to the *passive biped model* [McG90], seen as a hybrid oscillator. The passive biped model exhibits indeed a stable limit-cycle oscillation for appropriate parameter values that corresponds to periodic movements of the legs [SKN17]. The model has a continuous state variable $\boldsymbol{x}(t) = (\phi_1(t), \dot{\phi_1}(t), \phi_2(t), \dot{\phi_2}(t))^\top$. The dynamics is described by $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ with:

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} \dot{\phi_1} \\ \sin(\phi_1 - \gamma) \\ \dot{\phi_2} \\ \sin(\phi_1 - \gamma) + \dot{\phi_1}^2 \sin\phi_2 - \cos(\phi_1 - \gamma)\sin\phi_2 \end{pmatrix} \tag{3}$$

$$Reset(\boldsymbol{x}) = \begin{pmatrix} -\phi_1 \\ \dot{\phi_1}\sin(2\phi_1) \\ -2\phi_1 \\ \dot{\phi_1}\cos 2\phi_1(1 - \cos 2\phi_1) \end{pmatrix} \tag{4}$$

$$Guard(\boldsymbol{x}) \equiv (2\phi_1 - \phi_2 = 0 \ \wedge \phi_2 < -\delta). \tag{5}$$

We set $\delta = 0.1$ and $\gamma = 0.009$. See [McG90] for details. We give in Fig. 5 a typical cyclic trajectory in plans ϕ_1 and ϕ_2, during one period T. The coordinates of the parallelepiped vertices are for plan ϕ_1:

$((0.067939, -0.083172), (0.067943, -0.083172), (0.067943, -0.083169), (0.067939, -0.083169))$,

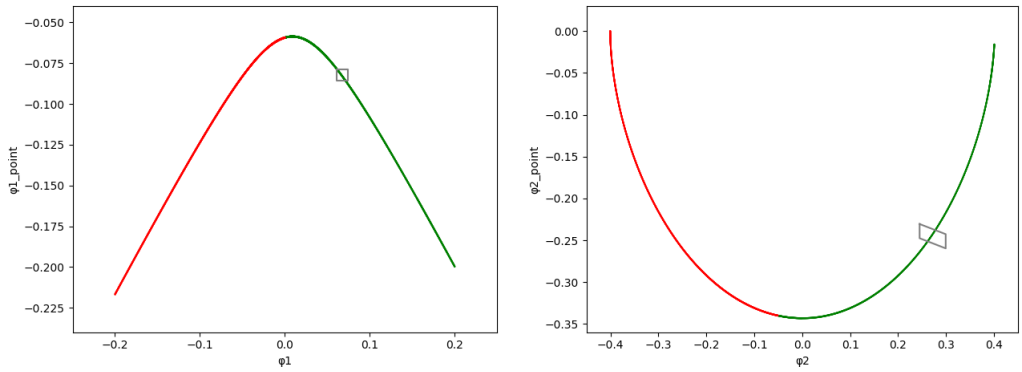

Figure 5: Biped: A cyclic trajectory for plan ϕ_1 (left) and ϕ_2 (right); the green zone indicates the contractive area ($\lambda < 0$) and the red zone the expansive one ($\lambda > 0$)

and for plan ϕ_2:

$((0.271972, -0.242725), (0.271983, -0.242734), (0.271983, -0.242731), (0.271972, -0.242722))$.

These parallepipeds are depicted in Fig. 6 (and also at magnified scale in Fig. 5). The time-step used in Euler's method is $\tau = 2 \cdot 10^{-5}$. The period of the system is $T = 776440\tau$. The radius expansion factor after one period is $E = 2.63$. The number of periods considered for synchronization is $k = 30$, and $\epsilon = 0.1$.

Fig. 6 depicts 10 (pairs of) initial balls with centers located on the parallelepiped *perimeters*, both in plan ϕ_1 and ϕ_2. The coordinates of these 10 (pairs of) centers, given under the form $(\phi_1, \dot{\phi_1}, \phi_2, \dot{\phi_2})$, are:

$((0.067940, -0.083172, 0.27198, -0.242729), (0.067942, -0.083168, 0.271975, -0.242727),$
$(0.067941, -0.083168, 0.271973, -0.242723), (0.067943, -0.0831719, 0.271978, -0.242727),$
$(0.067940, -0.0831682, 0.271973, -0.242726), (0.067941, -0.0831719, 0.271981, -0.242732),$
$(0.067940, -0.0831682, 0.271979, -0.242731), (0.067942, -0.0831719, 0.271976, -0.242725),$
$(0.067943, -0.0831682, 0.271977, -0.242729), (0.067941, -0.0831719, 0.271981, -0.242730))$

The coordinates $(\phi'_1, \dot{\phi'_1}, \phi'_2, \dot{\phi'_2})$ of their images after 30 periods are:

$((0.0679418, -0.0831697, 0.271978, -0.242729), (0.0679434, -0.0831707, 0.271983, -0.242732),$
$(0.0679425, -0.0831712, 0.271982, -0.242732), (0.0679416, -0.0831713, 0.271979, -0.242729),$
$(0.0679412, -0.0831698, 0.271976, -0.242726), (0.0679408, -0.0831702, 0.271976, -0.242726),$
$(0.0679431, -0.0831701, 0.271981, -0.242730), (0.0679407, -0.0831703, 0.271976, -0.242726),$
$(0.0679426, -0.0831700, 0.271980, -0.242729), (0.0679405, -0.0831707, 0.271977, -0.242729))$

The two components $(\phi_1, \dot{\phi_1})$ and $(\phi_2, \dot{\phi_2})$ of an initial point, as well as the two components $(\phi'_1, \dot{\phi'_1})$ and $(\phi'_2, \dot{\phi'_2})$ of its image, are all the 4 represented with the same color in Fig. 6. The CPU time taken for computing the 10 images is 6,800 seconds (for a program[5] written in Python running on the same machine used for the Brusselator example). Table 2 gives the phases of the 10 (pairs of) points shown in Fig. 6. After $k = 30$ periods, we have $\Delta(phase(centers)) \leqslant 0.25$. Since $\min(e_1/f_1, e_2/f_2) \approx 0.15$, the difference of phase between the components of a point starting *anywhere* from a ball (not necessarily fom its center), becomes always $\leqslant 0.4$. Here again, the proof has been done for 10 balls, but should be done for the whole set of balls covering S.

Table 2: The list of phases of 10 ball centers in the biped example.

Phases						
Point	Phase initial point in ϕ_1	Phase initial point in ϕ_2	Phase image point in ϕ_1	phase image point in ϕ_2	Δ(phase (centers)) for initial point	Δ(phase (centers)) for image point
1	0.88	0.29	0.45	0.48	0.59	0.03
2	0.38	0.75	0.05	0.02	0.37	0.03
3	0.55	0.94	0.27	0.07	0.39	0.21
4	0.14	0.48	0.52	0.35	0.34	0.17
5	0.88	0.94	0.62	0.64	0.05	0.03
6	0.55	0.20	0.71	0.65	0.35	0.06
7	0.72	0.39	0.14	0.23	0.33	0.09
8	0.30	0.71	0.74	0.67	0.40	0.07
9	0.22	0.61	0.25	0.32	0.40	0.08
10	0.72	0.16	0.78	0.53	0.56	0.25

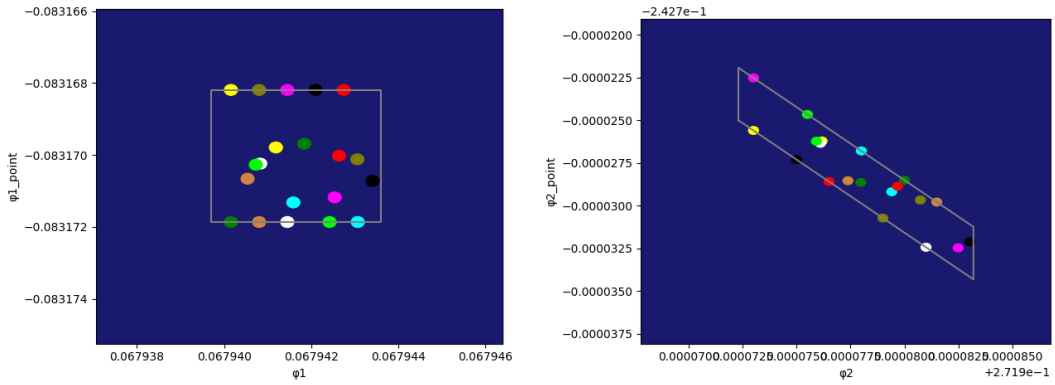

Figure 6: Biped: Synchronization of 10 (pairs of) balls, located initially on the parallelogram perimeters, after $k = 30$ periods (without radius expansion for clarity).

6 Final Remarks

We have described a symbolic reachability method to prove phase synchronization of oscillators, and illustrated it on the Brusselator and biped examples. The method is inspired by the classical "direct method" which shows that a *finite* number of points, displaced from their original position on a synchronization orbit, return after some time into a close neighborhood of the orbit. In contrast to the classical method, our symbolic method shows an analogous property for the *infinite* set S of points located around a portion of the orbit. Such a set S can be determined using simulation methods, but we assume here that it is given. Note that our method guarantees that the solution components are *almost* synchronized when they pass into S, whereas standard synchronization states the stronger property of *convergence* to the synchronization orbit.

Because of the magnification of the balls on a non-contractive space ($\lambda > 0$), one is forced to start with small initial balls, and the coverage of S requires a priori a huge number of balls. However, as explained on the Brusselator example, the analysis can be decomposed into periods, and accessibility per period proven *separately* from one intermediate area to the next, thus exponentially decreasing the number of balls. Note that the ball magnification problem does *not* occur on a *contractive* system ($\lambda < 0$), e. g., for Brusselator with a *large diffusion coefficient* σ, so the reachability analysis is easier in this case.

We focused here on $n = 2$ components with state space dimension $m = 2$. The extension to $n, m \geqslant 3$ is easy in principle, but causes combinatorial explosion of the number of balls covering S. In order to solve this "curse of dimensionality", it would be interesting in future work to adapt the classical "adjoint" method (or *phase reduction* [SKN17]) rather than the "direct" method used here.

We outlined an approach in order to solve the synchronization problem. The symbolic Euler's method has been used here for convenience, but could be replaced by any other symbolic reachability procedure. Given a subset $S = S_1 \times S_2$ appropriately selected within the state space, and a covering of S made of a set of couples of balls, the objective is to show that the elements of each couple return to S periodically, getting closer from each other at each time. We have shown in this paper that this property holds for the Brusselator and biped examples for a subset of the covering of S. The challenge is to show the property for all the couples of the covering, which requires to increase the power of the current tools of reachability analysis.

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