# Bilinear Boundary Optimal Control of a Kirchhoff Plate Equation

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August 7, 2021

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16th May, 2021

**Abstract**

We consider the problem of optimal control of a Kirchhoff plate equation governed by bilinear control such as coefficient like $hz$. The question is to obtain a distributed control which minimizes a function cost constituted of the deviation between a desired state and the reached one, and the energy term. The purpose of this study is to prove that an optimal control exists, and it is characterized as a solution to an optimality system. Thus, we give a sufficient condition for the uniqueness of such a control.

**Keywords:**
Kirchhoff plate equation, Boundary bilinear control, Optimal control problem.

## 1 Introduction

In this paper, we are concerned with the following system,

$$\begin{cases} z_{tt} + \Delta^2 z = 0 & Q = \Omega \times ]0, T[, \\ \Delta z + (1-\mu) B_1 z = 0 & \Sigma = \Gamma \times ]0, T[, \\ \dfrac{\partial}{\partial \upsilon} \Delta z + (1-\mu) B_2 z = k z_t + h z & \Sigma = \Gamma \times ]0, T[, \\ z(x,y,0) = z_0,\, z_t(x,y,0) = z_1 & \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is an open, bounded domain with boundary $\Gamma$ and $\upsilon = (n_1, n_2)$ denotes an outward unit normal to the boundary, the operators $B_1$ and $B_2$ are given by

$$B_1 z = 2 n_1 n_2 z_{xy} - n_1^2 z_{yy} - n_2^2 z_{xx},$$

$$B_2 z = \frac{\partial}{\partial \tau} [(n_1^2 - n_2^2) z_{xy} + n_1 n_2 (z_{yy} - z_{xx})],$$

where $\tau = (-n_2, n_1)$ denotes tangential direction. In the boundary conditions, $0 < \mu < \dfrac{1}{2}$ is the Poisson's ratio and $k > 0$ is a positive constant.
The objective functional is given by

$$J(h) = \frac{1}{2}\left( \int_Q (z - z_d)^2 dQ + \beta \int_\Sigma h^2 d\Sigma \right). \tag{2}$$

where $z$ is a solution of System $(1)$, $z_d \in L^2(\Omega)$ is a desired value and $\beta$ is a constant positive. The cost of implementing the control belonging

$$U = \{h \in L^\infty(\Gamma) / -M \leq h \leq M\} \tag{3}$$

where $M$ is a constant positive such that $0 < M < 2k - 1$.

We consider the following optimal control problem:

$$J(h^*) = \min_{h \in U} J(h), \tag{4}$$

Our purpose is to establish the existence and uniqueness of solutions for the System (1), seek an optimal control $h^* \in U$ satisfying the optimal control problem (4), and derive some necessary optimality conditions for the optimal control $h^*$.

In recent years, plate models and control have received great attention by researches, we can refer to Lagnese and Lions [9] , Lagnese, Leugering and Schmidt [11] and Li and Yong [14]. They have successful applications in many disciplines, namely, economics, environment,management and engineering etc. In the context of control theory and, in particular, optimal control problem, some authors have studied a variety of plate models. In [8] the authors have discussed an optimal control of a Kirchoff plate with boundary control that allows to minimize a functional cost which contains the energy of the control and the gradient of the error between the actual trajectory and the desired. In [17], R. Prakash has studied an optimal control problem for the time-dependent Kirchhoff-Love plate with distributed control. Also in [2] the authors have considered a control problem for the system of non-linear Karman's equations for a thin elastic plate. Existence and uniqueness of an optimal control have been established. On the other hand, the bilinear control problem for the Kirchoff plate equation has been first studied by Bradely and Lenhart [5] on what concerns a bilinear spatial control problem. Also, in collaboration with Young [6] they have considered a bilinear system excited by spatial-temporal controls, and in [4] Bradely and Lenhart have considered the same problem with distributed control. In [16] the authors have studied a Von Karman plate equation and have derived the existence and uniqueness of a spatial bilinear optimal control which is a function of the spatial variables.
Other papers have examined bilinear boundary control problems, which is the case of Lenhart and Wilson

[13] which have studied the optimal control of an equation with the convective boundary condition such as the bilinear control that represents a heat transfer coefficient. The used approach consists in finding a unique optimal control in terms of the solution of an optimality system, while Zerrik and El Kabouss [18] have discussed a bilinear boundary control problem of an output of parabolic systems with bounded control set. Moreover, in [19], they have studied an optimal control problem for the heat equation in order to give a control that leads to a state as close possible to a desired state, only on a subregion of the domain of evolution under unbounded controls sets.

The bilinear optimal control of a Kirchhoff Plate equation was considered in ([4], [6], [5]) with internal control, so in this paper we extend to the case of boundary control using the approach given in ([13], [19]), we prove existence of an optimal control solution for Problem (4) by a minimizing sequence argument. Then we derive the necessary optimality conditions of the optimal control and we prove that the optimal control is unique for small time T.

An outline of the remainder of the paper is as follows: In Section 2, we prove existence of an optimal control solution of Problem (4). In Section 3, we obtain a characterization of an optimal control as a solution of an optimality system which is derived by differentiating the cost functional with respect to the control, and we discuss a condition uniqueness.

## 2 Existence of a boundary optimal control

In this section, we show that System (5) has a unique solution. Then we show existence of an optimal control solution for (4) by a minimizing sequence argument.
We consider the following system

$$\begin{cases} z_{tt}+\Delta^2 z = f & Q=\Omega\times]0,T[,\\ \Delta z+(1-\mu)B_1 z=0 & \Sigma=\Gamma\times]0,T[,\\ \dfrac{\partial}{\partial \upsilon}\Delta z+(1-\mu)B_2 z = kz_t+hz+g & \Sigma=\Gamma\times]0,T[,\\ z(.,0)=z_0,\, z_t(.,0)=z_1 & \Omega, \end{cases} \tag{5}$$

where $f\in L^2(Q)$ and $g\in L^2(\Sigma)$.
Let us first define the bilinear form.

$$a(z,w)=\int_\Omega \left[\Delta z\Delta w+(1-\mu)(2z_{xy}w_{xy}-z_{yy}w_{xx}-z_{xx}w_{yy})\right]d\Omega \tag{6}$$

We know that ([10]):

$$m_1\|z\|^2_{H^2(\Omega)} \le a(z,z) \le m_2\|z\|^2_{H^2(\Omega)} \qquad \text{for all } z\in H^2(\Omega), \tag{7}$$

where $m_1$ and $m_2$ are positive constants.
Also, we define

$$b(z,w) = \int_\Gamma \big(kz_t w + hz_t w + gw\big) d\Gamma, \tag{8}$$

For notational convenience, we set

$$\begin{aligned} H &= H^2(\Omega) \times L^2(\Omega) \\ z &= z(h), \quad \tilde{z} = (z, z_t) \end{aligned}$$

We present our definition of weak solution:

**Definition 1** *Given $h \in U$, we say that a function $z \in C\big([0,T]; H^2(\Omega)\big)$, with $z_t \in C\big([0,T]; L^2(\Omega)\big)$, $z_{tt} \in C\big([0,T]; (H^2(\Omega))'\big)$ is a weak solution of the problem (1) satisfies:*

$$\int_0^T \langle z_{tt}, w\rangle dt + \int_0^T a(z,w)(t)dt + \int_0^T b(z,w)(t)dt = \int_Q f w dQ, \qquad for\ all \quad w \in H^2(\Omega) \tag{9}$$

*for any $w \in H^2(\Omega)$ and $0 \le t \le T$.*
*$z(0) = z_0$, $z_t(0) = z_1$, $\langle .,. \rangle$ denotes the duality pairing of $\big[H^2(\Omega)\big]'$ and $H^2(\Omega)$.*

By using the technique from [3], [7] and [12], we represent System (5) as an abstract ordinary differential equation, we define the operator $\mathscr{A} : L^2(\Omega) \longrightarrow L^2(\Omega)$ by

$$\mathscr{A} z = \Delta^2 z, \quad for \quad z \in D(\mathscr{A}) = \Big\{ z \in H^2(\Omega) : \Delta z + (1-\mu) B_1 z|_\Gamma = 0 \quad \text{and} \quad \frac{\partial}{\partial \upsilon}\Delta z + (1-\mu) B_2 z|_\Gamma = 0 \Big\}.$$

with $\mathscr{A}^*$ is adjoint operator. We also make use of the Green maps $G : L^2(\Gamma) \longrightarrow L^2(\Omega)$, defined by :

$$\begin{cases} G g_1 = z \Longleftrightarrow \Delta^2 z = 0 & in \quad \Omega, \\ \Delta z + (1-\mu) B_1 z = 0 & \text{on} \quad \Gamma, \\ \dfrac{\partial}{\partial \upsilon}\Delta z + (1-\mu) B_2 z = g_1 & \text{on} \quad \Gamma. \end{cases} \tag{10}$$

with $G^*$ is adjoint operator. It can be shown that (see [15]):

$$G \in \mathscr{L}(H^s(\Gamma) \longrightarrow H^{7/2+s}(\Omega)), \quad for\, all\, s \in \mathbb{R}. \tag{11}$$

Next we define : $A : D(A) \subset H \longrightarrow H$ by

$$A\tilde{z} = \begin{pmatrix} 0 & I \\ \mathscr{A} & 0 \end{pmatrix} \begin{pmatrix} z \\ z_t \end{pmatrix} \qquad \text{with } D(A) = D(\mathscr{A}) \times H^2(\Omega). \tag{12}$$

Note that $A$ is skew adjoint, hence $D(A) = D(A^*)$ and $A$ generates a unitary group on $H$ [3].

We define another linear operator $\mathscr{B} : L^2(\Gamma) \longrightarrow D(A)'$ $\big($see [3], page 8 with $\mathscr{U} = L^2(\Gamma)$ $\big)$

$$\mathscr{B}w = \begin{pmatrix} 0 \\ \mathscr{A}Gw \end{pmatrix}, \tag{13}$$

The adjoint operator $\mathscr{B}^*$ of $\mathscr{B}$ is computed by:

$$(\mathscr{B}w, \tilde{z})_{H^2(\Omega)\times L^2(\Omega)} = (w, G^*\mathscr{A}^* z_2)_{L^2(\Gamma)} \quad \text{ for } \tilde{z} \in D(A^*) \, and \, w \in L^2(\Gamma).$$

We obtain

$$\mathscr{B}^*\tilde{z} = G^*\mathscr{A}^* z_2, \quad \text{ for } \tilde{z} \in D(A). \tag{14}$$

On the other hand, by using the Green formula, we can show (see [3]) that

$$G^*\mathscr{A}^* z = -z\,|_\Gamma, \qquad for \, z \in D(\mathscr{A}). \tag{15}$$

Next $\mathscr{F}$ is linear operator from $H^2(\Omega) \times L^2(\Omega) \longrightarrow L^2(\Gamma)$ given by

$$\mathscr{F}\tilde{z} = -k\mathscr{B}^*\tilde{z}. \tag{16}$$

where $\tilde{z} = (z_1, z_2)$. By the above notation, Equation (5) may be written as an abstract ODE

$$\begin{aligned} \frac{d}{dt}\tilde{z}(t) &= A\tilde{z}(t) + \mathscr{B}\mathscr{F}\tilde{z}(t) + \mathscr{B}\mathscr{H}(\tilde{z}(t)) + \tilde{B}(\tilde{z}(t)), \\ \tilde{z}(0) &= (z_0, z_1), \end{aligned} \tag{17}$$

where $\mathscr{H}(\tilde{z}(t)) = hz + g$, and $\tilde{B}(\tilde{z}(t)) = \begin{pmatrix} 0 \\ \\ f \end{pmatrix}$.

Now we prove that System (5) has a unique weak solution $z(t)$ in $C([0;T];H)$, for this we need the following results:

**Lemma 2.1**
*Let* $A_{\mathscr{F}} = A + \mathscr{B}\mathscr{F}$ *with domain* $D(A_F) = \{\tilde{x} = (x_1, x_2) \in H^2(\Omega) \times L^2(\Omega) : x_2 \in H^2(\Omega)$ *and* $x_1 + kGG^*\mathscr{A}x_2 \in D(\mathscr{A})\}$.

$(H1)$ $A_{\mathscr{F}}$ *generate an exponentially stable semigroup on* $H^2(\Omega) \times L^2(\Omega)$.

$(H2)$ $A_{\mathscr{F}}^{-1}\mathscr{B} \in \mathscr{L}(L^2(\Gamma) \longrightarrow H^2(\Omega) \times L^2(\Omega))$.

$(H3)$ $\int_0^T \|\mathscr{B}^* e^{A_{\mathscr{F}}^* t}\tilde{x}\|_{L^2(\Gamma)} dt \leq C\|\tilde{x}\|_{H^2(\Omega)\times L^2(\Omega)}$ *for some* $0 < T < \infty$ *and for all* $\tilde{x} \in D(A_F^*)$.

**Proof** See Theorem 3.1 $\big(\text{with } \tilde{\mathscr{B}}_1 F_1 = 0\big)$ [3].

**Lemma 2.2** *[3]*
*Assume (H1) and (H2) hold. Then*

$$\sup_{0\leq t\leq T} \|\int_0^t e^{A_{\mathscr{F}}(t-r)}\mathscr{B}z(r))dr\|_{H^2(\Omega)\times L^2(\Omega)} \leq C\|z\|_{C([0,T];L^2(\Gamma))}, \tag{18}$$

**Theorem 1**
*For* $\tilde{z}_0 = (z_0, z_1) \in H = H^2(\Omega) \times L^2(\Omega)$ *and* $h \in U$*, System (5) has a unique weak solution* $\tilde{z} = (z, z_t)$ *in* $C([0;T];H)$.

**Proof** The solution $z$ to (5) can be written as:

$$\tilde{z}(t) = e^{A_{\mathscr{F}}t}\tilde{z}_0 + \int_0^t e^{A_{\mathscr{F}}(t-r)}\mathscr{B}\mathscr{H}(z(r))dr + \int_0^t e^{A_{\mathscr{F}}(t-r)}\tilde{B}(\tilde{z}(r))dr,$$

We prove that the map $T_h$

$$T_h\tilde{z}(t) = e^{A_{\mathscr{F}}t}\tilde{z}_0 + \int_0^t e^{A_{\mathscr{F}}(t-r)}\mathscr{B}\mathscr{H}(\tilde{z}(.,r))dr + \int_0^t e^{A_{\mathscr{F}}(t-r)}\tilde{B}(\tilde{z})(.,r)dr.$$

has a unique fixed point in $C([0,T_0];H)$.

Step 1. We prove that, if $T_0$ is small enough, there exists a unique fixed point such that

$$T_h\tilde{z}(t) = \tilde{z}(t) \qquad in\ C([0,T_0];H)$$

To use the contraction mapping theorem, we need to show that $T_h$ is bounded and contractive.
For boundedness,

$$\|T_h\tilde{z}\|_{C([0;T_0];H)} = \|e^{A_{\mathscr{F}}t}\tilde{z}_0 + \int_0^t e^{A_{\mathscr{F}}(t-r)}\mathscr{B}\mathscr{H}(\tilde{z}(.,r))dr + \int_0^t e^{A_{\mathscr{F}}(t-r)}\tilde{B}(\tilde{z})(.,r)dr\|_{C([0;T_0];H)}$$

$$\leq \|e^{A_{\mathscr{F}}t}\tilde{z}_0\|_{C([0;T_0];H)} + \sup_{0\leq t\leq T_0}\int_0^t \|e^{A_{\mathscr{F}}(t-r)}\mathscr{F}\mathscr{H}(\tilde{z}(.,r))\|_H dr + \sup_{0\leq t\leq T_0}\int_0^t \|e^{A_{\mathscr{F}}(t-r)}\tilde{B}(\tilde{z})(.,r)\|_H dr.$$

Using Lemma (2.2), we get

$$\|T_h\tilde{z}\|_{C([0;T_0];H)} \leq \|e^{A_{\mathscr{F}}t}\tilde{z}_0\|_H + C\sup_{0\leq t\leq T_0}\int_0^t \|h(.,r)z(.,r)+g(.,r)\|_{L^2(\Gamma)}dr + \sup_{0\leq t\leq T_0}\int_0^t \|e^{A_F(t-r)}f(.,r)\|_{L^2(\Omega)}dr.$$

Since $A_{\mathscr{F}}$ generates an exponentially stable semi-group on $H^2(\Omega)\times L^2(\Omega)$, we obtain

$$\begin{aligned}
\|T_h\tilde{z}\|_{C([0;T_0];H)} &\leq C\|\tilde{z}_0\|_H + C\sup_{0\leq t\leq T_0}\int_0^t \|h(.,r)z(.,r)\|_{L^2(\Gamma)}dr + CT_0\|g\|_{C([0,T_0];L^2(\Gamma)} + CT_0\|f\|_{C([0,T_0];L^2(\Omega))} \\
&\leq C\|\tilde{z}_0\|_H + C\sup_{0\leq t\leq T_0}\int_0^t \|h(.,r)\|_{L^\infty(\Gamma)}\|z(.,r)\|_{L^2(\Gamma)}dr + CT_0\|g\|_{C([0,T_0];L^2(\Gamma)} + CT_0\|f\|_{C([0,T_0];L^2(\Omega))}.
\end{aligned}$$

Since $\|h(.,r)\|_{L^\infty(\Gamma)} \leq M$, we have

$$\|T_h\tilde{z}\|_{C([0;T_0];H)} \leq C\|\tilde{z}_0\|_H + MC\sup_{0\leq t\leq T_0}\int_0^t \|z(.,r)\|_{L^2(\Gamma)}dr + CT_0\|g\|_{C([0,T_0];L^2(\Gamma))} + CT_0\|f\|_{C([0,T_0];L^2(\Omega))}.$$

For $n\geq 2$, there exists a constant $C>0$ such that for every $z\in H^1(\Omega)$ see [1]

$$\|z\|_{L^2(\partial\Omega)} \leq C\|z\|_{H^1(\Omega)},$$

it yields

$$\|T_h\tilde{z}\|_{C([0;T_0];H)} \leq C\|\tilde{z}_0\|_H + MC\sup_{0\leq t\leq T_0}\int_0^t \|z(.,r)\|_{H^1(\Omega)}dr + CT_0\|g\|_{C([0,T_0];L^2(\Gamma))} + CT_0\|f\|_{C([0,T_0];L^2(\Omega))}.$$

Hence,

$$\|T_h\tilde{z}\|_{C([0;T_0];H)} \leq C\|\tilde{z}_0\|_H + MC\sup_{0\leq t\leq T_0}\int_0^t \|z(.,r)\|_{H^2(\Omega)}dr + CT_0\|g\|_{C([0,T_0];L^2(\Gamma))} + CT_0\|f\|_{C([0,T_0];L^2(\Omega))}.$$

Which gives

$$\|T_h\tilde{z}\|_{C([0;T_0];H)} \leq C\|\tilde{z}_0\|_H + CMT_0\|\tilde{z}\|_{C([0;T_0];H^2(\Omega)\times L^2(\Omega))} + CT_0\|g\|_{C([0,T_0];L^2(\Gamma))} + CT_0\|f\|_{C([0,T_0];L^2(\Omega))}.$$

We obtain that $T_h$ is bounded.
For contraction, :
similarly for any $\tilde{z}=(z_1,z_2), \tilde{w}=(w_1,w_2)\in C([0;T_0];H)$

$$\|T_h\tilde{z}-T_h\tilde{w}\|_{C([0;T_0];H)} \leq CMT_0\|\tilde{z}-\tilde{w}\|_{C([0;T_0];H)}.$$

Taking $T_0 < \dfrac{1}{MC}$, we obtain that $T_h$ is a contractive mapping for $t \leq T_0$. Thus, we have existence of a unique fixed point on $C([0;T_0];H)$.

Step 2. Extend the above result to a solution on $[T_0, 2T_0]$. We set $\tilde{z}(T_0)$ as the new initial data. By a second contraction argument, we have a unique solution on $C([T_0;2T_0];H)$. Repeating the process a finite number of times, yields the result on $[0,T]$.

$\square$

We now present a priori estimate needed for existence of an optimal control.

**Lemma 2.3**
*if $\tilde{z}_0 = (z_0, z_1) \in H = H^2(\Omega) \times L^2(\Omega)$, $g \in L^2(\Sigma)$ and $f \in L^2(Q)$. Then, the weak solution $z$ of System (5) satisfies the following inequalities:*

$$\begin{aligned} &\sup_{0\leq t\leq T}\Big(\|z(t)\|_{H^2(\Omega)}+\|z_t(t)\|_{L^2(\Omega)}\Big)+\|z_t\|_{L^2(0,T,L^2(\Gamma))}+\|z_{tt}\|_{L^2(0,T,(H^2(\Omega))')} \\ &\qquad \leq C\Big(\|\tilde{z}_0\|_H+\|g\|_{L^2(0,T;L^2(\Gamma))}+\|f\|_{L^2(0,T;L^2(\Omega)}\Big), \end{aligned} \tag{19}$$

**Proof**
Multiplying System (5) by $z_t$ and integrating over $Q = \Omega \times [0,t]$, we obtain

$$\int_Q \left(z_{tt}z_t + \Delta^2 z z_t - f z_t\right) dQ = 0.$$

Hence,

$$\frac{1}{2}\int_0^t \frac{d}{dt}\Big(\|z_t\|^2_{L^2(\Omega)} + \|\Delta z\|^2_{L^2(\Omega)}\Big)dt + \int_\Sigma (1-\mu)\left[B_1 z \frac{\partial}{\partial \upsilon} z_t - B_2 z z_t\right] d\Sigma + \int_\Sigma \left(kz_t^2 + hzz_t + gz_t\right) d\Sigma = \int_0^t \int_\Omega f z_t \, d\Omega dt$$

Using Lemma 4.1 (see [3]), we obtain

$$\begin{aligned} \frac{1}{2}\int_0^t \frac{d}{dt}\Big(\|z_t\|^2_{L^2(\Omega)} + \|\Delta z\|^2_{L^2(\Omega)}\Big)dt + (1-\mu)\int_0^t\int_\Omega \left[2z_{xy}z_{xy_t} - z_{yy}z_{xx_t} - z_{xx}z_{yy_t}\right] d\Omega dt + \int_\Sigma \left(kz_t^2 + hzz_t + gz_t\right) d\Sigma \\ = \int_0^t \int_\Omega f z_t \, d\Omega dt \end{aligned}$$

Therefore, we can obtain the following from (6):

$$\frac{1}{2}\int_0^t \frac{d}{dt}\|z_t\|^2_{L^2(\Omega)}dt + \frac{1}{2}\int_0^t \frac{d}{dt} a(z,z) dt + \int_\Sigma \left(kz_t^2 + hzz_t + gz_t\right) d\Sigma = \int_0^t \int_\Omega f z_t \, d\Omega dt.$$

We get

$$\|z_t(t)\|^2_{L^2(\Omega)} + a(z(t),z(t)) + 2\int_\Sigma \left(kz_t^2 + hzz_t + gz_t\right) d\Sigma = \|z_1\|^2_{L^2(\Omega)} + a(z_0,z_0) + 2\int_0^t \int_\Omega f z_t \, d\Omega dt$$

Applying the Young inequality, we obtain

$$\|z_t\|^2_{L^2(\Omega)} + a(z(t),z(t)) + 2k\int_\Sigma z_t^2 d\Sigma \le \|z_1\|^2_{L^2(\Omega)} + a(z_0,z_0) + M\|z_t\|^2_{L^2(\Sigma)} + M\|z\|^2_{L^2(\Sigma)} + \|g\|^2_{L^2(\Sigma)} + \|z_t\|^2_{L^2(\Sigma)} + \int_0^t \|f\|^2_{L^2(\Omega)} dt + \int_0^t \|z_t\|^2_{L^2(\Omega)} dt$$

Since dim $\Omega = 2$, We know by Sobolev imbeddings that $H^1(\Omega) \subset L^p(\Gamma)$ for any $p > 1$ (see Theorem 5.22 of [1]) and consequently

$$\|z_t\|^2_{L^2(\Omega)} + a(z(t),z(t)) + (2k - M - 1)\int_\Sigma z_t^2 d\Sigma \le \|z_1\|^2_{L^2(\Omega)} + a(z_0,z_0) + \|g\|^2_{L^2(\Sigma)} + M\int_0^t \|z\|^2_{H^1(\Omega)} dt + \int_0^t \|f\|^2_{L^2(\Omega)} dt + \int_0^t \|z_t\|^2_{L^2(\Omega)} dt$$

Using (7), we have

$$\|z_t\|^2_{L^2(\Omega)} + m\|z\|^2_{H^2(\Omega)} + (2k - M - 1)\|z_t\|^2_{L^2(\Sigma)} \le C\|\tilde{z}_0\|^2_H + \|g\|^2_{L^2(0,T,L^2(\Gamma))} + \|f\|^2_{L^2(0,T,L^2(\Omega))} + M\int_0^t \|z\|^2_{H^2(\Omega)} dt + \int_0^t \|z_t\|^2_{L^2(\Omega)} dt$$

Hence,

$$\|z_t\|^2_{L^2(\Omega)} + m\|z\|^2_{H^2(\Omega)} + (2k - M - 1)\|z_t\|^2_{L^2(\Sigma)} \le \|\tilde{z}_0\|^2_H + \|g\|^2_{L^2(0,T,L^2(\Gamma))} + \|f\|^2_{L^2(0,T,L^2(\Omega))} + C\int_0^t \|\tilde{z}\|^2_{H^2(\Omega)\times L^2(\Omega)} dt \tag{20}$$

Since $2k - M - 1 > 0$, we obtain

$$\|\tilde{z}\|^2_{H^2(\Omega)\times L^2(\Omega)} \le C\left(\|g\|^2_{L^2(0,T,L^2(\Gamma))} + \|f\|^2_{L^2(0,T,L^2(\Omega))} + \|\tilde{z}_0\|^2_H\right) + C\int_0^t \|\tilde{z}\|^2_{H^2(\Omega)\times L^2(\Omega)} dt$$

Using Gronowall's inequality, we obtain

$$\|\tilde{z}\|^2_{H^2(\Omega)\times L^2(\Omega)} \le C\left(\|g\|^2_{L^2(0,T,L^2(\Gamma))} + \|f\|^2_{L^2(0,T,L^2(\Omega))} + \|\tilde{z}_0\|^2_H\right)$$

Taking the supremum, this gives

$$\sup_{0\leq t\leq T}\left(\|z(t)\|_{H^2(\Omega)}+\|z_t(t)\|_{L^2(\Omega)}\right)\leq C\Bigg(\|g\|^2_{L^2(0,T,L^2(\Gamma))}+\|f\|^2_{L^2(0,T,L^2(\Omega))}+\|\tilde{z}_0\|^2_H\Bigg) \tag{21}$$

From (20) and (21), we can obtain

$$\sup_{0\leq t\leq T}\Bigg(\|z(t)\|_{H^2(\Omega)}+\|z_t(t)\|_{L^2(\Omega)}\Bigg)+\|z_t\|_{L^2(0,T,L^2(\Gamma))}\leq C\Bigg(\|\tilde{z}_0\|_H+\|g\|_{L^2(0,T;L^2(\Gamma))}+\|f\|_{L^2(0,T;L^2(\Omega)}\Bigg), \tag{22}$$

Now we prove the inequality

$$\|z_{tt}\|_{L^2(0,T,(H^1(\Omega))')}\leq C\Bigg(\|z(0)\|_{H^1(\Omega)}+\|z_t(0)\|_{L^2(\Omega)}+\|f\|_{L^2(0,T,L^2(\Omega))}+\|g\|_{L^2(0,T,L^2(\Gamma))}\Bigg). \tag{23}$$

In view of (9), we infer that

$$\text{for } w\in H^2(\Omega),\quad |\langle z_{tt},w\rangle|\leq|a(z,w)|+|b(z,w)|+|\int_\Omega fwdx|. \tag{24}$$

Together with (7) and (8), we obtain

$$\begin{aligned}
|a(z,w)|+|b(z,w)| &\leq m_2\|z\|_{H^2(\Omega)}\|w\|_{H^2(\Omega)}+k\|z_t\|_{L^2(\Gamma)}\|w\|_{L^2(\Gamma)}+M\|z\|_{L^2(\Gamma)}\|w\|_{L^2(\Gamma)}+\|g\|_{L^2(\Gamma)}\|w\|_{L^2(\partial\Omega)}\\
&\leq m_2\|z\|_{H^2(\Omega)}\|w\|_{H^2(\Omega)}+k\|z_t\|_{L^2(\Gamma)}\|w\|_{H^1(\Omega)}+M\|z\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}+\|g\|_{L^2(\Gamma)}\|w\|_{H^1(\Omega)}\\
&\leq\Bigg(m_2\|z\|_{H^2(\Omega)}+k\|z_t\|_{L^2(\Gamma)}+M\|z\|_{H^2(\Omega)}+\|g\|_{L^2(\Gamma)}\Bigg)\|w\|_{H^2(\Omega)}\\
&\leq C\Bigg(\|z\|_{H^2(\Omega)}+\|z_t\|_{L^2(\Gamma)}+\|g\|_{L^2(\Gamma)}\Bigg)\|w\|_{H^2(\Omega)}.
\end{aligned}$$

Where C is a constant. We can deduce the following,

$$|\langle z_{tt},w\rangle|\leq C\Bigg(\|z\|_{H^2(\Omega)}+\|z_t\|_{L^2(\Gamma)}+\|g\|_{L^2(\Gamma)}+\|f\|_{L^2(\Omega)}\Bigg)\|w\|_{H^2(\Omega)},$$

From inequality (21) and (22), we can obtain

$$\|z_{tt}\|_{L^2(0,T,(H^1(\Omega))')}\leq C\Bigg(\|z(0)\|_{H^2(\Omega)}+\|z_t(0)\|_{L^2(\Omega)}+\|f\|_{L^2(0,T,L^2(\Omega))}+\|g\|_{L^2(0,T,L^2(\Gamma))}\Bigg).$$

$\square$

Now we obtain existence of an optimal control.

**Theorem 2**
*There exists an optimal control $h^* \in U$ minimizing the objective functional $J(h)$ for $h \in U$.*

**Proof**
Let $h_n \in U$ be a minimizing sequence such that

$$\lim_{n\to\infty} J(h_n) = \inf_{h\in\mathscr{U}} J(h).$$

Denote $z_n = z(h_n)$. By lemma 2.3 we have

$$\begin{aligned}\sup_{0\leq t\leq T}\Big(\|z_n\|_{H^2(\Omega)}+\|(z_n)_t\|_{L^2(\Omega)}\Big)+\|(z_n)_t\|_{L^2(0,T,L^2(\Gamma))}+\|(z_n)_{tt}\|_{L^2(0,T,(H^2(\Omega))')}\\ \leq C\Big(\|z(0)\|_{H^2(\Omega)}+\|z_t(0)\|_{L^2(\Omega)}+\|g\|_{L^2(0,T;L^2(\Gamma))}+\|f\|_{L^2(0,T;L^2(\Omega)}\Big),\end{aligned} \tag{25}$$

By weak compactness, there exists $z^*$ in $C([0,T],H^2(\Omega))$ such that

$$\begin{aligned} z_n &\rightharpoonup z^* \quad \text{weakly* in } L^\infty([0,T],H^2(\Omega)),\\ \frac{\partial z_n}{\partial t} &\rightharpoonup \frac{\partial z^*}{\partial t} \quad \text{weakly* in } L^\infty([0,T],L^2(\Omega)),\\ \frac{\partial z_n}{\partial t} &\rightharpoonup \frac{\partial z_n^*}{\partial t} \quad \text{weakly in } L^2(\Gamma),\\ \frac{\partial^2 z_n}{\partial t^2} &\rightharpoonup \frac{\partial^2 z_n^*}{\partial t^2} \quad \text{weakly in } L^2([0,T],(H^2(\Omega))'),\\ h_n &\rightharpoonup h^* \text{ weakly in } L^2(\Gamma)\end{aligned}$$

The mapping $\Phi : H^2(\Omega) \longrightarrow \Phi_{|\Gamma} \in H^{3/2}(\Gamma)$ is continuous. Then

$$z_n \longrightarrow z^* \text{ in } H^{3/2}(\Gamma) \subset L^2(\Gamma).$$

By the definition of a weak solution, we have

$$\langle (z_n)_{tt}, w\rangle + a(z_n,w) + b(z_n,w) = \int_\Omega f w d\Omega, \qquad \text{for all} \quad w \in H^2(\Omega) \tag{26}$$

Since

$$z_n \longrightarrow z^* \quad \text{strongly in} \quad L^2(\Gamma) \tag{27}$$

$$h_n \rightharpoonup h^* \quad \text{weakly in} \quad L^2(\Gamma) \tag{28}$$

then

$$h_n z_n \longrightarrow h^* z^* \text{ in } L^2(\Gamma).$$

We may now pass to the limit as $n \longrightarrow \infty$ in the weak formulation of $z_n$ , we obtain

$$\langle z^*_{tt}, w\rangle + a(z^*, w) + b(z^*, w) = \int_\Omega f w d\Omega, \qquad \text{for all} \quad w \in H^2(\Omega) \tag{29}$$

Where $z^* = z(h^*)$ is the solution of System (5) with control $h^*$. Since

$$J(h^*) = \frac{1}{2}\int_0^T \|z^*(.,t) - z_d(.)\|^2_{L^2(\Omega)} dt + \frac{\beta}{2}\int_0^T \|h^*(.,t)\|^2_{L^2(\Gamma)} dt. \tag{30}$$

Using lower semi-continuity of $L^2$ norm with respect to weak convergence, we have

$$\begin{aligned}
J(h^*) &\leq \lim_{n\to\infty} \frac{1}{2}\int_0^T \|z_n(.,t) - z_d(.)\|^2_{L^2(\Omega)} dt + \lim_{n\to\infty} \inf \frac{\beta}{2}\int_0^T \|h_n(.,t)\|^2_{L^2(\Gamma)} dt \\
&\leq \lim_{n\to\infty} \inf J(h_n) \\
&= \inf_{h\in U} J(h).
\end{aligned}$$

Hence $h^*$ is an optimal control solution of Problem (5).

□

# 3 Characterization of an Optimal Control

To obtain a characterization of an optimal control, we derive an optimality system differentiating the objective functional $J(h)$ with respect to the control $h$. We must first prove the appropriate differentiability of the mapping

$$h \longrightarrow (z(h), z_t(h)) = \tilde{z}(h)$$

**Lemma 3.1**
*The mapping $h \in U \longrightarrow \tilde{z}(h) = \big(z(u), z_t(h)\big) \in C([0,T]; H^2(\Omega) \times L^2(\Omega))$ is differentiable in the following sense:*

$$\frac{z_{h+\varepsilon u} - z_h}{\varepsilon} \rightharpoonup \psi \quad \text{weakly* in } L^\infty([0,T];H),$$

*as $\varepsilon \to 0$, for any $h$ satisfying $h + \varepsilon u \in U$ for $\varepsilon$ small. Moreover $\tilde{\psi} = (\psi, \psi_t)$ is a weak solution of the following system:*

$$\begin{cases} \psi_{tt} + \Delta^2 \psi = 0 & Q = \Omega \times ]0,T[, \\ \Delta \psi + (1-\mu) B_1 \psi = 0 & \Sigma = \Gamma \times ]0,T[, \\ \dfrac{\partial}{\partial \upsilon} \Delta \psi + (1-\mu) B_2 \psi = k \psi_t + h \psi + uz & \Sigma = \Gamma \times ]0,T[, \\ \psi(.,0) = 0,\ \psi_t(.,0) = 0 & \Omega, \end{cases} \tag{31}$$

**Proof**
Let $z^\varepsilon = z(h + \varepsilon u)$ and $z = z(h)$, then $(z^\varepsilon - z)/\varepsilon$ is a weak solution of

$$\begin{cases} \left(\dfrac{z^\varepsilon - z}{\varepsilon}\right)_{tt} + \Delta^2 \left(\dfrac{z^\varepsilon - z}{\varepsilon}\right) = 0 & Q = \Omega \times ]0,T[, \\ \Delta \left(\dfrac{z^\varepsilon - z}{\varepsilon}\right) + (1-\mu) B_1 \left(\dfrac{z^\varepsilon - z}{\varepsilon}\right) = 0 & \Sigma = \Gamma \times ]0,T[, \\ \dfrac{\partial}{\partial \upsilon} \Delta \left(\dfrac{z^\varepsilon - z}{\varepsilon}\right) + (1-\mu) B_2 \left(\dfrac{z^\varepsilon - z}{\varepsilon}\right) = k \left(\dfrac{z^\varepsilon - z}{\varepsilon}\right)_t + h \left(\dfrac{z^\varepsilon - z}{\varepsilon}\right) + uz^\varepsilon & \Sigma = \Gamma \times ]0,T[, \\ \left(\dfrac{z^\varepsilon - z}{\varepsilon}\right)(.,0) = 0,\ \left(\dfrac{z^\varepsilon - z}{\varepsilon}\right)_t (.,0) = 0 & \Omega, \end{cases} \tag{32}$$

Using Lemma 2.3, we get

$$\begin{aligned} \sup_{0 \le t \le T} \left( \frac{(z^\varepsilon - z)(t)}{\varepsilon} \|_{H^2(\Omega)} + \| \frac{z_t^\varepsilon - z_t}{\varepsilon} \|_{L^2(\Omega)} \right) &\le C \| u z_\varepsilon \|_{L^2([0,T];L^2(\Gamma))} \\ &\le C \|u\|_{L^\infty(\Omega)} \sup_{0 \le t \le T} \|z_\varepsilon\|_{L^2(\Gamma)} \\ &\le C \|u\|_{L^\infty(\Omega)} \sup_{0 \le t \le T} \|z_\varepsilon\|_{H^1(\Omega)} \\ &\le C \|u\|_{L^\infty(\Omega)} \sup_{0 \le t \le T} \|z_\varepsilon\|_{H^2(\Omega)} \\ &\le C_0 \big( \|z(0)\|_{H^2(\Omega)} + \|z_t(0)\|_{L^2(\Omega)} \big), \end{aligned} \tag{33}$$

where $C_0$ is independent of $\varepsilon$.
With similar arguments to (33), we can deduce

$$\|\frac{\partial^2}{\partial t^2}\big(\frac{z^\varepsilon - z}{\varepsilon}\big)\|_{L^2(0,T,(H^2(\Omega))')} \leq C_1\big(\|z(0)\|_{H^2(\Omega)} + \|z_t(0)\|_{L^2(\Omega)}\big),$$

and

$$\|\frac{\partial}{\partial t}\big(\frac{z^\varepsilon - z}{\varepsilon}\big)\|_{L^2(0,T,(L^2(\Gamma))} \leq C_2\big(\|z(0)\|_{L^2(\Omega)} + \|z_t(0)\|_{L^2(\Omega)}\big).$$

where $C_1$ and $C_2$ are independent of $\varepsilon$. Hence, on a subsequence as $\varepsilon \longrightarrow 0$, we have:

$$\frac{z^\varepsilon - z}{\varepsilon} \rightharpoonup \psi \quad \text{weakly* in } L^\infty([0,T];H).$$

In a similar way to the proof of Theorem (2), we can obtain that $\psi$ is the weak solution of System (31).

□

Now, we derive our optimality system.

**Proposition 1**
*Given an optimal control $h^* \in U$ and the corresponding $\tilde{z}_{h^*} = \tilde{z}(h^*) = (z, z_t)$, there exists a weak solution $\tilde{p} = (p, p_t)$ in $H$ to the adjoint problem:*

$$\begin{cases} p_{tt} + \Delta^2 p = z_{h^*} - z_d & Q = \Omega \times ]0,T[, \\ \Delta p + (1-\mu)B_1 p = 0 & \Sigma = \Gamma \times ]0,T[, \\ \dfrac{\partial}{\partial \upsilon}\Delta p + (1-\mu)B_2 p = -kp_t + h^* p & \Sigma = \Gamma \times ]0,T[, \\ p(.,T) = 0,\ p_t(.,T) = 0 & \Omega, \end{cases} \tag{34}$$

*Furthermore $h^* \in U$ satisfies*

$$h^*(.,t) = \max\left(-M, \min\Big(\frac{1}{\beta} z_{|\Sigma} p_{|\Sigma}, M\Big)\right).$$

**Remark 3.1**
*If we set*

$$\begin{aligned} q(x,y,t) &= p(x,y,T-t), \\ w(x,y,t) &= h(x,y,T-t), \\ f(x,y,t) &= z_h(x,y,T-t) - z_d, \end{aligned}$$

*then, the adjoint equation becomes*

$$\begin{cases} q_{tt}+\Delta^2 q = f & Q=\Omega\times]0,T[, \\ \Delta q+(1-\mu)B_1 q = 0 & \Sigma=\Gamma\times]0,T[, \\ \dfrac{\partial}{\partial \upsilon}\Delta q+(1-\mu)B_2 q = kq_t + wq & \Sigma=\Gamma\times]0,T[, \\ q(.,0)=0,\, q_t(.,0)=0 & \Omega, \end{cases} \tag{35}$$

*Therefore, it admits a unique solution.*

**Proof**

Let $h\in U$ be an optimal control and $\tilde{z}=\tilde{z}(h)$ be the corresponding optimal solution. Let $h+\varepsilon u\in U$ for $\varepsilon>0$ and $\tilde{z}^{\varepsilon}=\tilde{z}(h+\varepsilon u)$ be the corresponding weak solution of (5). We compute the directional derivative of the cost functional $J(h)$ with respect to $h$ in the direction of $u$. Since $J$ is supposed to attain its minimum for $h$, we have

$$\begin{aligned} 0 &\le \lim_{\varepsilon\to 0^+}\frac{J(h^*+\varepsilon u)-J(h^*)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0^+}\int_0^T \frac{1}{2\varepsilon}(\|z_\varepsilon - z_d\|^2_{L^2(\Omega)} - \|z_{h^*}-z_d\|^2_{L^2(\Omega)})dt + \frac{\beta}{2}\int_\Sigma (2uh^*+\varepsilon u^2)\,d\Sigma \\ &= \lim_{\varepsilon\to 0^+}\int_Q \left(\frac{z_\varepsilon - z_{h^*}}{\varepsilon}\right)\left(\frac{z_\varepsilon + z_{h^*}-2z_d}{2}\right)dQ + \frac{\beta}{2}\int_\Sigma (2uh^*+\varepsilon u^2)\,d\Sigma \\ &= \int_Q \psi(z_{h^*}-z_d)dQ + \beta\int_\Sigma h^* u\,d\Sigma. \end{aligned}$$

where $\psi$ is a solution of System (31) corresponding to $h^*$.

Substituting in the adjoint equation (34) for $(z_{h^*}-z_d)$ from (34), we obtain

$$\begin{aligned} 0 &\le \int_Q \psi(z_{h^*}-z_d)dQ + \beta\int_\Sigma h^* u\,d\Sigma \\ &= \int_Q \psi\big(p_{tt}+\Delta^2 p\big)dQ + \beta\int_\Sigma h^* u d\Sigma \\ &= \int_0^T \left(\langle \psi_{tt}, p\rangle + \int_0^T a(\psi,p)dt - \int_\Gamma (kp_t - hp)\psi dx\right)dt + \beta\int_\Sigma h^* u d\Sigma. \end{aligned} \tag{36}$$

Multiplying System (31) by $p$ and integrating over $(0,T)$, we have

$$\int_0^T \left(\langle \psi_{tt}, p\rangle + \int_0^T a(\psi,p)dt + \int_\Gamma (k\psi_t + h\psi)p dx\right)dt = -\int_\Sigma uzp d\Sigma.$$

Integrating by parts gives

$$\int_0^T \left( \langle \psi_{tt}, p \rangle + \int_0^T a(\psi, p)dt - \int_\Gamma kp_t \psi dx + \int_\Gamma hp\psi dx \right) dt = -\int_\Sigma uzpd\Sigma.$$

It follows

$$\int_0^T \langle \psi_{tt}, p \rangle dt + \int_0^T a(\psi, p)dt = \int_\Gamma kp_t \psi d\Gamma - \int_\Gamma hp\psi dx - \int_\Sigma uzpd\Sigma. \tag{37}$$

Then from (36) and (37), we obtain

$$0 \leq -\int_\Sigma uzpd\Sigma + \beta \int_\Sigma h^* ud\Sigma. \tag{38}$$

Taking $u = \max\Big(-M, \min\Big(\frac{1}{\beta} z_{|\Sigma} p_{|\Sigma}, M\Big)\Big) - h^*$, then $u(h^* - \frac{1}{\beta} z_{|\Sigma} p_{|\Sigma})$ is negative and

$$\left( \max\Big(-M, \min\Big(\frac{1}{\beta} z_{|\Gamma} p_{|\Sigma}, M\Big)\Big) - h^* \right) \left( h^* - \frac{1}{\beta} z_{|\Sigma} p_{|\Sigma} \right) = 0$$

So,

if $M \leq \frac{1}{\beta} z_{|\Sigma} p_{|\Sigma}$ we have $(M - h^*)(h^* - \frac{1}{\beta} z_{|\Sigma} p_{|\Sigma}) = 0$, then (since $h^* \leq M$),

$$h^* = M,$$

if $-M \leq \frac{1}{\beta} z_{|\Sigma} p_{|\Sigma} \leq M$, we have $(\frac{1}{\beta} z_{|\Sigma} p_{|\Gamma} + h^*)(h^* - \frac{1}{\beta} z_{|\Sigma} p_{|\Sigma}) = 0$ and then

$$h^* = \frac{1}{\beta} z_{|\Sigma} p_{|\Sigma},$$

if $-M \geq \frac{1}{\beta} z_{|\Sigma} p_{|\Sigma}$, we have $(-M - h^*)(h^* - \frac{1}{\beta} z_{|\Sigma} p_{|\Sigma}) = 0$ and then (since $h^* \geq -M$)

$$h^* = -M.$$

Finally, we conclude that

$$h^* = \max\Big(-M, \min\Big(\frac{1}{\beta} z_{|\Sigma} p_{|\Sigma}, M\Big)\Big).$$

This completes the proof.

$\square$

# 4 Uniqueness of the Optimal Control

In this section, we formulate sufficient conditions for the uniqueness of the optimal control solution of Problem (3).

**Proposition 2** *Suppose the solution of Systems (5) and (34) is bounded on Q. For T sufficiently small, the solution of Problem (3) is unique.*

**Proof**
Suppose we have two weak solutions corresponding to two optimal controls, $h$ and $\bar{h}$:

$$\tilde{z} = (z, z_t), \qquad \hat{z} = (\bar{z}, \bar{z}_t)$$

We then have that $\tilde{z} - \hat{z}$ satisfies the following equation

$$\begin{cases} (z-\bar{z})_{tt} + \Delta^2(z-\bar{z}) = 0 & Q = \Omega\times]0,T[, \\ \Delta(z-\bar{z}) + (1-\mu)B_1(z-\bar{z}) = 0 & \Sigma = \Gamma\times]0,T[, \\ \dfrac{\partial}{\partial \upsilon}\Delta(z-\bar{z}) + (1-\mu)B_2(z-\bar{z}) = k(z-\bar{z})_t + h(z-\bar{z}) + (h-\bar{h})\bar{z} & \Sigma = \Gamma\times]0,T[, \\ (z-\bar{z})(.,0) = 0,\ (z-\bar{z})_t(.,0) = 0 & \Omega, \end{cases} \tag{39}$$

Where we denote

$$h = \max\Big(-M, \min\Big(\frac{1}{\beta} z_{|\Sigma} p_{|\Sigma}, M\Big)\Big). \tag{40}$$

and

$$\bar{h} = \max\Big(-M, \min\Big(\frac{1}{\beta} \bar{z}_{|\Sigma} \bar{p}_{|\Sigma}, M\Big)\Big). \tag{41}$$

It can be shown by direct computation that

$$|h-\bar{h}| \leq \frac{1}{\beta}\big(|p-\bar{p}||\bar{z}| + |z-\bar{z}||p|\big). \tag{42}$$

Applying Lemma (19) to System (39), we

$$\sup_{0\leq t\leq T} \|(z-\bar{z})(t)\|_{H^2(\Omega)} \leq C\|(h-\bar{h})\bar{z}\|_{L^2(\Sigma)} \tag{43}$$

Therefore, from (42), we can deduce the following:

$$\|(h-\bar{h})\bar{z}\|^2_{L^2(\Sigma)} = \int_\Sigma |h-\bar{h}|^2|\bar{z}|^2 d\Sigma \leq \frac{1}{\beta^2}\int_\Sigma \big(|p-\bar{p}||\bar{z}| + |z-\bar{z}||p|\big)^2 |\bar{z}|^2 d\Sigma.$$

It follows

$$\begin{aligned}\int_\Sigma |h-\bar{h}|^2|\bar{z}|^2 d\Sigma &\le \frac{2}{\beta^2}\int_\Sigma \left(|p-\bar{p}|^2|\bar{z}|^2+|z-\bar{z}|^2|p|^2\right)|\bar{z}|^2 d\Sigma,\\ &\le \frac{2}{\beta^2}\int_\Sigma |p-\bar{p}|^2|\bar{z}|^4+|z-\bar{z}|^2|p|^2|\bar{z}|^2 d\Sigma,\\ &\le \frac{2}{\beta^2}\int_\Sigma |p-\bar{p}|^2|\bar{z}|^4 d\Sigma+\frac{1}{\beta^2}\int_\Sigma \left(|z-\bar{z}|^2|\bar{z}|^4+|z-\bar{z}|^2|p|^4\right)d\Sigma,\end{aligned}$$

By using Holder's inequality, we obtain

$$\begin{aligned}\int_\Sigma |h-\bar{h}|^2|\bar{z}|^2 d\Sigma \le \frac{2}{\beta^2}\int_0^T &\left(\int_\Gamma |p-\bar{p}|^4 d\Gamma\right)^{1/2}\left(\int_\Gamma |\bar{z}|^8 d\Gamma\right)^{1/2} dt\\ +\frac{1}{\beta^2}\int_0^T &\left(\int_\Gamma |z-\bar{z}|^4 d\Gamma\right)^{1/2}\left(\int_\Gamma |\bar{z}|^8 d\Gamma\right)^{1/2} dt\\ +\frac{1}{\beta^2}\int_0^T &\left(\int_\Gamma |p|^8 d\Gamma\right)^{1/2}\left(\int_\Gamma |z-\bar{z}|^4 d\Gamma\right)^{1/2} dt.\end{aligned}$$

We get

$$\begin{aligned}\int_\Sigma |h-\bar{h}|^2|\bar{z}|^2 d\Sigma \le \frac{2}{\beta^2}\int_0^T &||p-\bar{p}||^2_{L^4(\Gamma)}||\bar{z}||^4_{L^8(\Gamma)} dt\\ +\frac{1}{\beta^2}\int_0^T &||z-\bar{z}||^2_{L^4(\Gamma)}||\bar{z}||^4_{L^8(\Gamma)} dt\\ +\frac{1}{\beta^2}\int_0^T &||p||^4_{L^8(\Gamma)}||z-\bar{z}||^2_{L^4(\Gamma)} dt.\end{aligned}$$

We use Holder's inequality, we have

$$\int_{\Sigma} |h-\bar{h}|^2 |\bar{z}|^2 d\Sigma \leq \frac{2}{\beta^2}\left[\int_0^T ||p-\bar{p}||^4_{L^4(\Gamma)} dt\right]^{1/2}\left[\int_0^T ||\bar{z}||^8_{L^8(\Gamma)} dt\right]^{1/2}$$

$$+\frac{1}{\beta^2}\left[\int_0^T ||z-\bar{z}||^4_{L^4(\Gamma)} dt\right]^{1/2}\left[\int_0^T ||\bar{z}||^8_{L^8(\Gamma)} dt\right]^{1/2}$$

$$+\frac{1}{\beta^2}\left[\int_0^T ||p||^8_{L^8(\Gamma)} dt\right]^{1/2}\left[\int_0^T ||z-\bar{z}||^4_{L^4(\Gamma)} dt\right]^{1/2}.$$

Since

$$\int_0^T ||p-\bar{p}||^4_{L^4(\Gamma)} dt \leq T \sup_{0\leq t\leq T} \|p-\bar{p}\|^4_{L^4(\Gamma)}, \qquad \int_0^T ||z-\bar{z}||^4_{L^4(\Gamma)} dt \leq T \sup_{0\leq t\leq T} \|z-\bar{z}\|^4_{L^4(\Gamma)}$$

$$\int_0^T ||p||^8_{L^8(\Gamma)} dt \leq T \sup_{0\leq t\leq T} \|p\|^8_{L^8(\Gamma)}, \qquad \int_0^T ||\bar{z}||^8_{L^8(\Gamma)} dt \leq T \sup_{0\leq t\leq T} \|\bar{z}\|^8_{L^8(\Gamma)}$$

We obtain

$$\int_{\Sigma} |h-\bar{h}|^2 |\bar{z}|^2 d\Sigma \leq \frac{2}{\beta^2} T \sup_{0\leq t\leq T} \|p-\bar{p}\|^2_{L^4(\Gamma)} \sup_{0\leq t\leq T} \|\bar{z}\|^4_{L^8(\Gamma)}$$

$$+\frac{1}{\beta^2} T \sup_{0\leq t\leq T} \|z-\bar{z}\|^2_{L^4(\Gamma)} \sup_{0\leq t\leq T} \|\bar{z}\|^4_{L^8(\Gamma)}$$

$$+\frac{1}{\beta^2} T \sup_{0\leq t\leq T} \|z-\bar{z}\|^2_{L^4(\Gamma)} \sup_{0\leq t\leq T} \|p\|^4_{L^8(\Gamma)}.$$

By using Sobolev imbeddings (see [1])

$$H^1(\Omega) \hookrightarrow L^p(\Gamma) \qquad \text{for any} \quad p>1,$$

We get

$$\int_{\Sigma}|h-\bar{h}|^2|\bar{z}|^2 d\Sigma \leq \frac{2}{\beta^2}T\sup_{0\leq t\leq T}\|p-\bar{p}\|^2_{H^1(\Omega)}\sup_{0\leq t\leq T}\|\bar{z}\|^4_{H^1(\Omega)}$$

$$+\frac{1}{\beta^2}T\sup_{0\leq t\leq T}\|z-\bar{z}\|^2_{H^1(\Omega)}\sup_{0\leq t\leq T}\|\bar{z}\|^4_{H^1(\Omega)}$$

$$+\frac{1}{\beta^2}T\sup_{0\leq t\leq T}\|z-\bar{z}\|^2_{H^1(\Omega)}\sup_{0\leq t\leq T}\|p\|^4_{H^1(\Omega)}.$$

It follows

$$\int_{\Sigma}|h-\bar{h}|^2|\bar{z}|^2 d\Sigma \leq \frac{2}{\beta^2}T\sup_{0\leq t\leq T}\|p-\bar{p}\|^2_{H^2(\Omega)}\sup_{0\leq t\leq T}\|\bar{z}\|^4_{H^2(\Omega)}$$

$$+\frac{1}{\beta^2}T\sup_{0\leq t\leq T}\|z-\bar{z}\|^2_{H^2(\Omega)}\sup_{0\leq t\leq T}\|\bar{z}\|^4_{H^2(\Omega)}$$

$$+\frac{1}{\beta^2}T\sup_{0\leq t\leq T}\|z-\bar{z}\|^2_{H^2(\Omega)}\sup_{0\leq t\leq T}\|p\|^4_{H^2(\Omega)}.$$

Since $p$, $z$, $\bar{p}$ and $\bar{z}$ are bounded in $H^2(\Omega)$, assuming that they are all bounded by a constant $C_1$ we obtain

$$\int_{\Sigma}|h-\bar{h}|^2|\bar{z}|^2 d\Sigma \leq \frac{C_1 T}{\beta^2}\left(\sup_{0\leq t\leq T}\|p-\bar{p}\|^2_{H^2(\Omega)}+\sup_{0\leq t\leq T}\|z-\bar{z}\|^2_{H^2(\Omega)}\right). \tag{44}$$

Combining (43) and (44), we deduce that

$$\sup_{0\leq t\leq T}\|z-\bar{z}\|^2_{H^2(\Omega)} \leq \frac{C_2 T}{\beta^2}\left(\sup_{0\leq t\leq T}\|p-\bar{p}\|^2_{H^2(\Omega)}+\sup_{0\leq t\leq T}\|z-\bar{z}\|^2_{H^2(\Omega)}\right). \tag{45}$$

On the other hand, $\hat{p}=p-\bar{p}$ is the solution of the following system

$$\begin{cases} (p-\bar{p})_{tt}+\Delta^2(p-\bar{p})=z-\bar{z} & Q=\Omega\times]0,T[,\\ \Delta(p-\bar{p})+(1-\mu)B_1(p-\bar{p})=0 & \Sigma=\Gamma\times]0,T[,\\ \dfrac{\partial}{\partial\upsilon}\Delta(p-\bar{p})+(1-\mu)B_2(p-\bar{p})=-k(p-\bar{p})_t+h(p-\bar{p})+(h-\bar{h})\bar{p} & \Sigma=\Gamma\times]0,T[,\\ (p-\bar{p})(.,T)=0,\ (p-\bar{p})_t(.,T)=0 & \Omega, \end{cases} \tag{46}$$

Using (19), we get

$$\sup_{0\le t\le T}\|(p-\bar p)(t)\|^2_{H^2(\Omega)} \le C\bigg(\|(z-\bar z)(t)\|^2_{L^2(Q)} + \|(h-\bar h)\bar p\|^2_{L^2(\Sigma)}\bigg). \tag{47}$$

In a similar way, to estimate $\|(h-\bar h)\bar z\|^2_{L^2(\Sigma)}$, we obtain

$$\|(h-\bar h)\bar p\|^2_{L^2(\Sigma)} \le \frac{C_3 T}{\beta^2}\bigg(\sup_{0\le t\le T}\|p-\bar p\|^2_{H^2(\Omega)} + \sup_{0\le t\le T}\|z-\bar z\|^2_{H^2(\Omega)}\bigg). \tag{48}$$

Hence, from (47) and (48), we have

$$\sup_{0\le t\le T}\|(p-\bar p)(t)\|^2_{H^2(\Omega)} \le T\big(C+\frac{C_4}{\beta^2}\big)\bigg(\sup_{0\le t\le T}\|p-\bar p\|^2_{H^2(\Omega)} + \sup_{0\le t\le T}\|z-\bar z\|^2_{H^2(\Omega)}\bigg). \tag{49}$$

Combining (49) and (45), we infer

$$\sup_{0\le t\le T}\|z-\bar z\|^2_{H^2(\Omega)} + \sup_{0\le t\le T}\|p-\bar p\|^2_{H^2(\Omega)} \le T\big(C+\frac{C_5}{\beta^2}\big)\bigg(\sup_{0\le t\le T}\|p-\bar p\|^2_{H^2(\Omega)} + \sup_{0\le t\le T}\|z-\bar z\|^2_{H^2(\Omega)}\bigg).$$

If we choose T small such that $T\big(C+\frac{C_5}{\beta^2}\big)<1$, then we obtain

$$\sup_{0\le t\le T}\|z-\bar z\|^2_{H^2(\Omega)} + \sup_{0\le t\le T}\|p-\bar p\|^2_{H^2(\Omega)} \le 0.$$

Which gives the uniqueness.

□

**Conclusion 1** *Optimal control of a class of Kirchhoff plate equation is considered in bounded bilinear boundary controls. Existence of an optimal control is proved and characterized by optimality conditions. A sufficient condition for uniqueness of the optimal control is given. We notice that questions are still open, for instance the case of optimal control of Kirchhoff equation with unbounded bilinear boundary controls.*

# References


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