Relationship Between K-Cutsets and Comb Inequalities

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1 Introduction

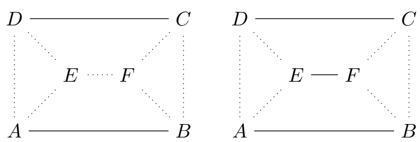
The TSP is the problem of finding a single cycle going through all the vertices of a graph such that the sum of the costs of the edges it contains is minimal. It has many applications and has been motivated by concrete problems, such as school bus routes, logistics, routing, etc. It is solved in CP with the WCC [2] and the k-cutset constraint [4]. The TSP can also be solved by the MIP dedicated solver Concorde [1] which uses an LP model in combination with the cutting-plane method. The CP and MIP use structural constraints for their solving. In this article, we show that the formulations look very different but they are in fact very similar.

2 The k-cutsets

Given $G = (X, O, M)$ a graph where X is a vertex set and O and M are two disjoint set representing optional and mandatory edges. In CP, we solve the problem consisting in finding a Hamiltonian cycle in G going through all the edges of M with the structural constraint of k-cutsets. This allows to delete some edges from O and add some edges to M. If there is a Hamiltonian cycle in G then for any K, a S-T cut of the graph G of cardinality k (i.e. $X = S + T$ and $S \cap T = \emptyset$ and K contains all the edges having one endpoint in S and one in T), then each of the $TSP(G)$ solutions takes an even and strictly positive number of K edges.

Thus, filtering rules can be deduced. If k is odd and K contains only mandatory edges then there is no solution. Otherwise, if $k = 2$ then both edges of K become mandatory, if k is odd with $k - 1$ mandatory edges then the optional edge of K is removed and if k is even with $k - 1$ mandatory edges then the optional edge of K becomes mandatory.

An example is given in the following Figure where solid lines are the edges of M and dotted lines are the edges of O. Given $K = \{(D, C), (E, F), (A, B)\}$. In the left graph, K has a cardinality 3 and have 2 mandatory edges $\{(D, C), (A, B)\}$, so (E, F) can be removed. In the right graph, K has a cardinality 3 and have 3 mandatory edges, then there is no solution.



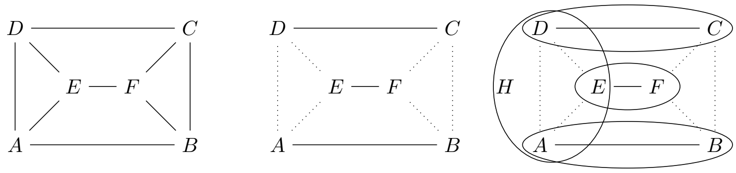
3 Comb inequalities

As stated in [3] : "A comb is defined by giving several subsets of nodes of the graph: We need one nonempty handle $H \subseteq V$, $H \neq V$ and $2k + 1$ pairwise disjoint, nonempty teeth $T_1, T_2, ..., T_{2k+1} \subseteq V$. for k at least 1. (So the number of teeth is odd and at least 3.) We also require each tooth to have at least one node in common with the handle and at least one node that is not in the handle."

The following cutting-plane is used in Concorde is based on the Comb inequality 1.

$$\sum_{e \in U(H)} x_e + \sum_{i=1}^{2k+1} x(T_i)) \leq |H| + k \tag{1}$$

An example is given in the following Figure. We note in 2 and 3 the solid lines = 1 and the dotted lines = 1/2 as a solution of the LP. The Comb inequality 1 is violated because $3+3 \leq 3+1$ is wrong, therefore the cutting-plane $\sum_{e \in U(H)} x_e + \sum_{i=1}^{2k+1} x(T_i)) \leq 4$ is generated.



1. Graph G 2. A feasible solution of G 3. Handle and teeth of G

4 Relationship

We show an original relationship between k-cutsets and Comb's inequalities.

LP generates a non-integer solution that is a lower bound of the TSP and looks for some S-T cuts in it such that the number of edges is odd, then the S-T cut is prohibited by the Comb inequality. Thus, the lower bound is potentially improved at each iteration.

CP removes edges from the graph, makes edges mandatory and looks in the current graph for all S-T cuts of size $k = 2, 3$ and apply filtering rules of the k-cutset.

Thus, the strategy of the LP is to look for some S-T cuts for generating cutting-plane, the strategy of the CP is to look for all S-T cuts where the cardinality of the cutset is 2 or 3. A very important remark is that finding all Comb inequalities is NP-Complete and finding k-cutsets for any k is NP-Complete. Each of the specific usages can be explained by the fact that the LP does not remove edges of the graph, so it will tend to have rather large cuts, the CP very quickly removes a large part of the edges of the graph, and will therefore have smaller cuts. Our experiments show that is rather hard to increase the size of k and having good computation times because of the complexity of the algorithm.

Experiments show in [1] that Comb's inequalities are essential for strong performance. Experiments show in [4] that k-cutsets allow to obtain gains of a factor 2. This experimentally confirms the importance of this type of cut for the resolution of TSP, beyond the method used.

To conclude, using the graph structure allows Concorde to become the best MIP solver for the TSP, similarly the k-cutset constraint leads to the best CP model for solving the TSP.

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