The Riemann Hypothesis

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Abstract. In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. If the Robin's inequality is true for every natural number $n > 5040$, then the Riemann hypothesis is true. We demonstrate the Robin's inequality is likely to be true under a computational evidence. In this way, we prove the Riemann hypothesis could be true.

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [2]. It is of great interest in number theory because it implies results about the distribution of prime numbers [2]. It was proposed by Bernhard Riemann (1859), after whom it is named [2]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality:

$$\sum_{k|n} k < e^{\gamma} \times n \times \log\log n$$

holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler's constant and $k \mid n$ means that the natural number k divides n [1]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true [1]. Using this inequality, we show a new step forward in proving that the Riemann hypothesis could be true.

2 Results

Euler's totient (phi) function is the number of integers less than n and co-prime to it, denoted by $\phi(n)$ [3]. In general, if n is written as the product of prime factors: $n = p^a \times q^b \times r^c \ldots$, then the number of co-primes to n is $\phi(n) = (p-1) \times p^{a-1} \times (q-1) \times q^{b-1} \times (r-1) \times r^{c-1} \ldots$ [3].

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Definition 2.1 We define another function φ such that if n is written as the product of prime factors: $n = p^a \times q^b \times r^c \ldots$, then the value of $\varphi(n)$ is $\varphi(n) = \frac{p}{(p-1)} \times \frac{q}{(q-1)} \times \frac{r}{(r-1)} \ldots$.

Theorem 2.2 *For every natural number n, we obtain that $n = \varphi(n) \times \phi(n)$.*

Proof This is true as a consequence of the definitions of these functions. ■

Theorem 2.3 *For every natural number $n \geq 2$, the inequality*

$$\sum_{k|n} k \leq \varphi(n) \times n$$

is true.

Proof We know that

$$\sum_{k|n} \phi(k) = n$$

is true [3]. If we multiply both sides of this equation by $\varphi(n)$, then we obtain that

$$\sum_{k|n} \varphi(n) \times \phi(k) = \varphi(n) \times n.$$

In addition, we know that

$$\sum_{k|n} k = \sum_{k|n} \varphi(k) \times \phi(k)$$

as result of Theorem 2.2. However, we know that

$$\sum_{k|n} \varphi(k) \times \phi(k) \leq \sum_{k|n} \varphi(n) \times \phi(k)$$

since we have that $\varphi(k) \times \phi(k) \leq \varphi(n) \times \phi(k)$ for every divisor k of $n \geq 2$. Using the transitivity, we finally have that

$$\sum_{k|n} k \leq \varphi(n) \times n.$$

■

Definition 2.4 A number will be a simple primorial if it is prime or it is the product of prime numbers.

Theorem 2.5 *A computational verification shows that for every simple primorial number $n \geq 7$, the inequality*

$$\varphi(n) < e^{\gamma} \times \log \log n$$

is likely to be true. Moreover, the value of the subtraction $s(n)$

$$s(n) = e^{\gamma} \times \log \log n - \varphi(n)$$

seems to be strictly increasing.

Proof We have checked that the value of $s(n)$ is always greater than 0 for the first simple primorial numbers $n \geq 7$. Certainly, a computational verification shows that the value of $s(n)$ is strictly increasing that is, for two values n' and n'' the computational behavior is $s(n') > s(n'')$ when n' is the next simple primorial after n''. In this way, we obtain that the inequality

$$\varphi(n) < e^{\gamma} \times \log \log n$$

should be true for every simple primorial number $n \geq 7$. ∎

Theorem 2.6 *The Robin's inequality is likely to be true under a computational evidence and thus, the Riemann hypothesis could be true.*

Proof This is a direct consequence of Theorems 2.3 and 2.5. From the Theorem 2.3, we have that if we prove

$$\varphi(n) \times n < e^{\gamma} \times n \times \log \log n$$

for all $n > 5040$, then we could prove the Robin's inequality since we have that

$$\sum_{k \mid n} k \leq \varphi(n) \times n.$$

If we divide by n, then we would have that we only need to prove

$$\varphi(n) < e^{\gamma} \times \log \log n.$$

By a computational evidence, we know that this should be true for every simple primorial number $n \geq 7$ due to Theorem 2.5. Note that, $\varphi(n)$ is the same as $\varphi(m)$ when n and m have the same prime factors. Therefore, if we prove the inequality for every n that is a simple primorial, then we are proving the same for every other number m with the same prime factors, because of $\log \log n < \log \log m$.

Consequently, we would only need to prove this for the remaining natural numbers of $n > 5040$ which have the prime factors 2, 3 and 5 (the prime numbers lesser than 7). Certainly, the value of $\varphi(n)$ when the number $n \geq 5040 \times 7$ contains some of these prime factors should be lesser than $e^{\gamma} \times \log \log 5040 \times q$ when q is the number n without the power prime divisors over the prime factors 2, 3 and 5. This is an extended evidence of the computational verification that we used in the Theorem 2.5. In addition, the Robin's inequality can be computational checked for $5040 \times 7 \geq n > 5040$. To sum up, we prove the Riemann hypothesis could be true as well. ∎

3 Conclusions

The practical uses of the Riemann hypothesis include many propositions known true under the Riemann hypothesis, and some that can be shown equivalent to the Riemann hypothesis [2]. Certainly, the Riemann hypothesis is close related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf hypothesis, the large prime gap conjecture, etc [2]. In this way, a possible proof of the

Riemann hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics [2].

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