Uncertainty Evaluation in Euler-Bernoulli and Timoshenko Bending Statics Problems

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Abstract. The error estimate of an adopted model is one of the main challenges in the quantification of uncertainty and in predictive science. For computational models of physical systems, the model inadequacy is frequently the major contributor to general predictive uncertainty. In stochastic structural mechanics, the uncertainties can be associated to the material and geometry as well as to the load on the structure and seeking out quantify the variability of the responses, generally associated to stresses and strains. Uncertainty is dealt with as a multivariate stochastic field where the system properties are modeled through their probability distribution. The Monte Carlo simulation emerges as a traditional model of reliability evaluation in order to solve the stochastic variational problem using finite elements, but, for more complex systems, the computing costs of this model becomes prohibitive. The proposal of the present work is to study the λ-Neumann method with a numerical strategy to quantify uncertainty when applied to the traditional Euler-Bernoulli and Timoshenko beam bending theory. The method is based on the Neumann series and, for problems of reaction-diffusion, bending of beams and plates presenting a satisfactory performance regarding the accuracy, a reduction in processing time and also in the non-intrusiveness of the computer program.

Keywords: structural stochastic mechanics; λ-Neumann method; beam theory.

1. Introduction

The technological evolution during recent decades is due in part to the development of mathematical and computational models that approach the real systems of engineering. The functionality of these models depends on the skill to predict suitable results with a high level of reliability in systems whose excitation and contour conditions often lack absolute control.

In structural stochastic mechanics the deterministic beam theories can be formulated from variational principles. The variational calculus is applied in Reddy (1984) where the problem is defined from a system of differential equations and the answers are obtained by approximation methods, among them the finite element method. On the other hand, the uncertainties have been

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associated with the stiffness coefficient and the load on beam and plates theories. This uncertainty arises from surface imperfections, material heterogeneity and seismic loading.

In Ghanem and Spanos (1991) the line of research in the stochastic analysis by finite elements basically presents two foci: the probabilistic modeling of the random mean and the numerical solution of the stochastic equations.

However, in general methods of obtainment of random variables assume that these variables obey a Gaussian probability distribution. The random parameters of the system are modeled within a second order stochastic process defined by its statistical moments, covariance function and cumulative functions or wave density spectrum. To obtain these estimators some methods are commonly adopted among them:

i. The *methods First-Order Moment* e *Second-Order Moment* found in Der-Kiureghian *et al*. (1987) is an algorithm that estimate the first and second order statistics through the transformation of statistical distribution spaces together with a finite element deterministic code.

ii. The *Perturbation method* expresses the stiffness matrix, the load and displacement nodal vector in terms of the Taylor expansion with respect to the random variables. In Nakagari and Hisada (1982) it is observed that, for first-order perturbations, the results present reasonable accuracy when dealing with small variabilities in the material properties.

iii. Response Surface Techniques are applied on the finite element deterministic formulation, where the optimum point search algorithm is expanded linear or quadratic function. It is meta model of simplified explicit function. Using Gaussian variables a number of finite element simulations are performed whose solution defines an approximation for a response surface.

iv. The most common and robust method is the *Monte Carlo simulation* studied in Shinozuka and Leone (1976), Shinozuka (1987) and Spanos and Mignolet (1986) being simple and flexible for parallel computing and without restrictions on the number of variables random. The Monte Carlo simulation is generally taken as a reference for checking other techniques.

v. The *Neumann series* is used to approximate the stiffness matrix inverse, this operation being a step in the solution of the system of equations in the finite element method. Yamazaki (1987), Yamazaki *et al*. (1988), Araújo and Awruch (1994) study the variability problem resulting from the spatial and material variability of the structure. Further applications of the Neumann expansion are found in Adomian and Malakian (1980), Shinozuka (1987), Chakraborty and Dey (1996), Chakraborty, and Bhattacharyya (2001).

In this last decade, an innovation adds improvements to the application of the Neumann series in the stochastic process by establishing a convergence parameter obtained from the structural optimization problem. The series is used to formulate a mathematical optimization problem whose objective function is defined by the search for the parameter λ. Computation time is reduced compared to the direct Monte Carlo method and the Neumann series using the first order for Neumann series and deterministic finite element analysis programs are easily applicable. The λ-Neumann Method was applied in Ávila and Beck (2015), Beck *et al* (2016) and Squarcio and Ávila (2017).

2. The Monte Carlo λ-Neumann Method

On the space of approximate solutions Avila *et al*. (2009) reports the consequences of uncertainty modeling on the solution process, indicating that for certain mechanical problems the use of a Gaussian process can lead to loss of coercion in the bi-linear way associated with the stochastic problem. The authors demonstrate that a Gaussian process to represent the uncertainty of system parameters can lead to non-convergence of the solution when applied to a problem of plate flexion with random parameters.

In structural reliability the random variable $\omega \in (\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is the σ-álgebra and P is the probability space. In this work we propose to model the random input variables as a uniform distribution where input variability is defined in the probability space and statistical moments are obtained for a kth realization of the random process. The space variable is defined in the set $x \in (0, l)$, where l is the beam length and the functional $u(x, \omega) \in L^2(\Omega, \mathcal{F}, P)$ is defined by the Hilbert space. The space of the results or outputs is obtained by the tensor product between the space of probability and the variational space, that is, the approximated solution space is constructed using isomorphism properties between spaces. The space must meet the Lax-Milgram Lemma where for isomorphism becomes approximately $L^2\big((\Omega, \mathcal{F}, P) \otimes H^2(0, l)\big)$. (Treves, 1967, Besold, 2000, apud Ávila et al, 2009 and Babuska *et al*., 2005).

From the point of view of stochastic structural mechanics, beam problems involve a differential equation with random coefficients representing the properties of the system. Thus, the variational stochastic formulation of the problem for the kth random sample becomes:

$$\begin{cases} \text{To determine } U(x, \omega_k) \in L^2\big((\Omega, \mathcal{F}, P) \otimes H^2(0, l)\big), \text{such that,} \\ K(x, \omega_k).U(x, \omega_k) = F(x, \omega_k), \\ + \text{ boundary conditions,} \end{cases} \tag{P.1}$$

where $U(x, \omega_k)$ is displacement vector.

It is considered the definite positive stiffness matrix $K(x, \omega_k)$, that is,

H1. $\exists \underline{\alpha}, \overline{\alpha} \in \mathbb{R}^+ \setminus \{0\}$ $\left|\underline{\alpha}, \overline{\alpha}\right| < \infty$, $P(\{\omega \in \Omega$: $K(x, \omega) \in [\underline{\alpha}, \overline{\alpha}]$, $\forall$ $x \in (0,l)\}=1)$,

and loading $F(x, \omega_k)$ with finite variance:

H2. $f \in L^2\big(\Omega, F, P | L^2(0, l)\big)$.

Hypothesis *H1* ensures that flexural stiffness coefficients are defined positive and uniformly limited in probability. Hypothesis *H2* guarantees the existence of the integral of $f: \mathbb{R}^n \to \mathbb{R}$. In this way, we obtain the random displacement vector with the inversion of the stiffness matrix such that:

$$U(x, \omega_k) = [K(x, \omega_k)]^{-1}.F(x, \omega_k). \tag{1}$$

For each realization of the random process it becomes necessary to obtain the solution of the linear system expressed by Eq. (1). In general, the computation of the inverse is not performed, since it involves a high computational cost and iterative methods such as Gauss, Jacobi, Gauss-Seidel, LU decomposition or Conjugated Gradients are used. Depending on the nature of the problem and the presence of nonlinearities, the number of random variables and the number of degrees of freedom solutions may become prohibitive. In this context the Neumann series becomes an alternative for estimating the random stiffness matrix.

On the other hand in the variational solution of differential equations, the differential equation is put into an equivalente variation form, and the approximate solution is assumed to be a combination $\sum \zeta_i \psi_i$ given approximation functions $\psi(x): D \to R^m$ and, the parameters $\zeta_i\big(K(\omega)\big): (\Omega, \mathcal{F}, P) \to \mathbb{R}^n$ are determined from the variational form. The approximation functions are derived using concepts from interpolation theory, and are therefore called interpolation functions. The finite element method can be interpreted as a piecewise application of the variational methods, among them the Galerkin method. The orthogonal polynomials of Hermite, Laguerre, Jacobi and Legendre represent a family of sub-spaces generated to be replaced in an ordinary differential equation. As a consequence, the inversion of the stiffness matrix becomes composed by the coefficients of these polynomials and the representation the uncertainty of the stiffness matrix, that is, the uncertainty function of parameter κ: $(0, l) \otimes \Omega \rightarrow \mathbb{R}^n$, is given to:

$$\kappa(x, \omega) = \mu_K(x) + \sum_{i=1}^{N} \psi_i(x)\zeta_i\left(K(\omega)\right), \tag{2}$$

where, $\mu_K(x)$ the first statistical moment of the stiffness matrix.

To perform this operation we must rewrite the equation of the problem representing the variability of the stiffness matrix by its statistical moments, $\Delta K\left(\xi(\omega_k)\right)$ and by the expected value of this matrix, $K_0(x)$ such that:

$$K\left(\xi_{\alpha\beta}(\omega_k)\right) = K_0(x) + \Delta K\left(\xi_{\alpha\beta}(\omega_k)\right), \tag{3}$$

where $\xi_{\alpha\beta}(\omega_k)$ are the random variables associated with the coefficients of the system of equations, α and β the elements of the stiffness matrix for the kth sample of the random process.

If the inverse of the stiffness matrix exists, that is, $\exists\ [K_0(x)]^{-1}$, such that the variability $\Delta\mathrm{K}\left(\xi_{\alpha\beta}(\omega_k)\right)$ around the expected value, $K_0(x)$ can be expressed by the argument of the Neumann series, $P\left(\xi_{\alpha\beta}(\omega_k)\right)$:

$$P\left(\xi_{\alpha\beta}(\omega_k)\right) = [K_0(x)]^{-1}\Delta K\left(\xi_{\alpha\beta}(\omega_k)\right). \tag{4}$$

The argument is a continuous linear operator $P: X \rightarrow X$, with isomorphism applied on a normalized space of X, such that $P^0 = I$, where I the identity matrix. This way we can rewrite Eq. (3), such that:

$$K\left(\xi_{\alpha\beta}(\omega_k)\right) = K_0(x)\left[I - P\left(\xi_{\alpha\beta}(\omega_k)\right)\right]. \tag{5}$$

Substituting Eq. (5) in the Eq. (1) we have to:

$$U\left(\xi_{\alpha\beta}(\omega_k)\right) = \left[K\left(\xi_{\alpha\beta}(\omega_k)\right)\right]^{-1} F\left(\xi_{\alpha\beta}(\omega_k)\right) = \left[I - P\left(\xi_{\alpha\beta}(\omega_k)\right)\right]^{-1} U_0(x), \tag{6}$$

where $U_0 = K_0.F$, such that, U_0:$(\Omega, \mathscr{F}, P) \rightarrow \mathbb{R}^n$ is the displacement obtained for the first statistical moment of the stiffness matrix.

Using the properties of the Neumann series and observe that, by definition of the identity matrix we have:

$$\left[K\left(\xi_{\alpha\beta}(\omega_k)\right)\right]^{-1} . K\left(\xi_{\alpha\beta}(\omega_k)\right) = K\left(\xi_{\alpha\beta}(\omega_k)\right) . \left[K\left(\xi_{\alpha\beta}(\omega_k)\right)\right]^{-1} = \mathrm{I}. \tag{7}$$

However, numerically this equality depends on the convergence of the series and the accuracy required in the programming. This approximation of the identity matrix defines the error related to the process and reveals how close the Neumann series approaches from the identity matrix. Using a first order approximation for the series, the strategy is based on formulating an optimization problem whose objective function is interpreted as the distance between matrix norms being defined by the convergence parameter λ represented by its components, λ_1 e λ_2, such that:

$$\left[I - P\left(\xi_{\alpha\beta}(\omega_k)\right)\right]^{-1} . \left[I - P\left(\xi_{\alpha\beta}(\omega_k)\right)\right] = \lambda_1 I + \lambda_2 \mathrm{P}\left(\xi_{\alpha\beta}(\omega_k)\right). \tag{8}$$

Using the first-order approximation of the inverse of the stiffness matrix, via Neumann series, defined an optimization problem whose objective function are finite-dimensional linear operators. In this way the following optimization problem can be established:

$$\begin{cases} \text{Find } (\lambda_1^*, \lambda_2^*) \in R^2 \text{ such that,} \\ (\lambda_1^*, \lambda_2^*) = \arg\min \left\{ \dfrac{1}{2} \left\| \lambda_1 [I - P(\xi_{\alpha\beta})] + \lambda_2 P(\xi_{\alpha\beta}) [I - P(\xi_{\alpha\beta})]^2 \right\| \right\}. \end{cases} \tag{P.2}$$

The objective function is nonnegative and convex, with the coordinates of the global optimum point $(\lambda_1^*, \lambda_2^*)$ obtained for stationarity condition, such that,

$$\nabla f(\lambda_1^*, \lambda_2^*) = 0. \tag{9}$$

Solving the system generated by Eq. (9) the convergence parameters are determined by:

$$\begin{cases} \lambda_1^* = \dfrac{\phi - \varphi \lambda_2^*}{\delta}, \\ \lambda_2^* = \dfrac{\delta\vartheta - \varphi\vartheta}{\delta\gamma - \vartheta^2}. \end{cases} \tag{10}$$

where:

$$\begin{cases} (\xi_{\alpha\beta}) = \{[I - P(\xi_{\alpha\beta})]U_0\}^t [I - P(\xi_{\alpha\beta})]U_0, \\ \varphi(\xi_{\alpha\beta}) = \{[I - P(\xi_{\alpha\beta})]U_0\}^t P(\xi_{\alpha\beta})[I - P(\xi_{\alpha\beta})]U_0, \\ \phi(\xi_{\alpha\beta}) = U_0^t [I - P(\xi_{\alpha\beta})]U_0, \\ (\xi_{\alpha\beta}) = \{P(\xi_{\alpha\beta})[I - P(\xi_{\alpha\beta})]U_0\}^t P(\xi_{\alpha\beta})[I - P(\xi_{\alpha\beta})]U_0, \\ \vartheta(\xi_{\alpha\beta}) = U_0^t P(\xi_{\alpha\beta})[I - P(\xi_{\alpha\beta})]U_0. \end{cases} \tag{11}$$

Thus, for the kth realization of the linear system the displacement vector is obtained with the linear approximation of the Neumann series, expressed by:

$$U_i(\lambda^*, \xi_{\alpha\beta}) = \left[\lambda_1 I + \lambda_2 P\left(\xi_{\alpha\beta}(\omega_k)\right)\right] U_i. \tag{12}$$

3. The Stochastic Problem of Bending and Rotation Beams

In Ávila and Beck (2010), there are differences between the stochastic Euler-Bernoulli and Timoshenko beam theories that establish uncertainty results associated to the elasticity module and to the second-order inertia moment. Despite the pronounced discrepancy among the results associated to uncertainty, the deterministic values are very close. It is important to note that the model was applied to beams with intermediate length.

3.1 – Euler-Bernoulli variational formulation

In the Euler-Bernoulli beam theory, also known as the classical beam theory, the cross section is assumed to remain plane and normal to the geometric axis. This hypothesis is equivalent to considering pure bending for which one has the variation uniform and symmetric normal stress distribution. The variational formulation can be found on Love (1944), Reddy (1984), Hugles (2000) and Ávila and Beck (2010).

The rotation of the cross section is represented by the angular displacement $\phi = dw/dx$ and beam bending problem is formulated for the kth sample of the random sample as follows:

$$\begin{cases} \text{To determine the vertical displacement } w(x,\omega_k) \in L^2\big((\Omega,\mathcal{F},P)\otimes H^m\big), \text{such that,} \\ \dfrac{d^2}{dx^2}\left[EI\dfrac{d^2w}{dx^2}\right](x,\omega_k) = q(x,\omega_k), \\ w(0,\omega) = w(l,\omega) = 0, \dfrac{dw}{dx}(0,\omega) = \dfrac{dw}{dx}(l,\omega_k) = 0, \forall x \in (0,l), \omega \in (\Omega,F,P), \end{cases} \tag{P.3}$$

where, *K(x,ω)=EI(x,ω)* is the flexural stiffness and $q(x,\omega_k)$ is limited loading with finite variance.

Employing numerical methods in the solution of problem is reduced problem restrictions and replaced the differential equation by a system of algebraic equations. Numerical solutions are originating from the Abstract Variational Problem.

To solve the proposed problem (P.3) it is considered v an approximation function such that $a(w,v), a: H \times H \to \mathbb{R}$ is a bilinear function, continuous and coercive and $\forall\, l = l(v) \in H^*$, a linear and continuous functional, such that,

$$\begin{cases} \text{For fixed } \xi_{\alpha\beta}(\omega_k), \text{find } w(x,\omega_k) \in L^2\big((\Omega,\mathcal{F},P)\otimes H^m\big), \text{such that,} \\ a(u,v) = l(v), \forall v(x) \in H. \end{cases} \tag{P.4}$$

where:

$$\begin{cases} a(u,v) = \displaystyle\int_\Omega \int_0^l K\frac{d^2w}{dx^2}\frac{d^2v}{dx^2}dx dP(\omega), \\ l(u) = \displaystyle\int_\Omega \int_0^l f v(x,\omega) dx dP(\omega). \end{cases} \tag{13}$$

Using the element finite method a set of approximation function samples $\{\psi_i(x)\}_{i=1}^N$ are applied on the terms of the Neumann series and the bilinear form can be write:

$$k_{ij}^0 = a_0(\psi_i,\psi_j) = \int_0^l \left(\mu_k \frac{d^2\psi_i}{dx^2}\frac{d^2\psi_j}{dx^2}\right)dx, \tag{14}$$

$$k_{ij}^q = a_q(\psi_i,\psi_j) = \int_0^l \left(\xi_q \frac{d^2\psi_i}{dx^2}\frac{d^2\psi_j}{dx^2}\right)dx. \tag{15}$$

Considering the representation of the stochastic system, Eq. (2) fixed to the kth sample of the random process, the problem becomes:

$$\begin{cases} \text{To determine } w(x,\omega_k) \in L^2\big((\Omega,\mathcal{F},P)\otimes H^m\big), \text{such that,} \\ \displaystyle\sum_{i=1}^{n} k_{ij}^0 w_i(\omega_k) + \xi(\omega_k)\sum_{i=1}^{n} k_{ij}^0 w_i(\omega_k) = l_{\omega_k}(v). \end{cases} \tag{P.5}$$

The proposal of the λ-Neumann method involves the assembly of the linear system through the expansion of Neumann, and the displacement vector is obtained from Eq. (12).

On the other hand, in the application of the unidirectional finite element method with two degrees of freedom per node the displacement vector is obtained from, for fixed $\xi_{\alpha\beta}(\omega_k)$:

$$w_{\omega_k}^{e^t} = \left\{ w_1(x) \quad \frac{dw_1}{dx}(x) \quad w_2(x) \quad \frac{dw_2}{d(x)}(x) \right\}. \tag{16}$$

The interpolation of vertical and angular displacement is performed separately for each of these variables. Since the displacements have two nodal values each is used the one-dimensional interpolation with two nodes. The displacement is interpolated by Hermite polynomials, that is,

$$U_{\omega_k} = N(\psi).w^e(x), \tag{17}$$

and the interpolation functions $N(\psi)$ are given by:

$$\begin{cases} N_1 = 1 - 3\psi^2(x) + 2\psi^3(x), \\ N_2 = l.[\psi - 2\psi^2(x) + \psi^3(x)], \\ N_3 = 3\psi^2(x) - 2\psi^3(x), \\ N_4 = l.[-\psi^2(x) + \psi^3(x)]. \end{cases} \tag{18}$$

The stochastic element stiffness matrix used for the Euler-Bernoulli beam is given by:

$$\left[K_{\omega_k}(x)\right] = \frac{EI_{\omega_k}(x)}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}. \tag{19}$$

And the nodal loading vector is obtained from the work done by the distributed forces

$$\left\{q_{\omega_k}(x)\right\}^e = \frac{q_{\omega_k}(x).l}{2}\left\{2 \quad \frac{l}{6} \quad 2 \quad -\frac{l}{6}\right\}. \tag{20}$$

The vertical displacement field is expressed in matrix form:

$$\begin{Bmatrix} w_{\omega_k}(x) \\ \phi_{\omega_k}(x) \end{Bmatrix} = \begin{bmatrix} [N_w(x)] & 0 \\ 0 & [N_\theta(x)] \end{bmatrix} \begin{Bmatrix} \underline{w}(x,\omega_k) \\ \underline{\phi}(x,\omega_k) \end{Bmatrix}^e, \tag{21}$$

where $N_w(x)$ e $N_\phi(x)$, interpolation functions, $\underline{w}(x,\omega_k)$ and $\underline{\phi}(x,\omega_k)$ the nodal displacement vectors.

3.2 – Timoshenko variational formulation

The hypothesis that the cross section remains normal the neutral line means to neglect the shearing strain and, in this sense, Timoshenko's beam bending theory considers that although the cross section remains plane simultaneously also an additional rotation occurs due to the shear stress, such that this section does not remain normal to the neutral line. The variational formulation can be found on Timoshenko (1921), Timoshenko (1922), Timoshenko and Goodier (1951), Mindlin and Deresiewicz (1954), Timoshenko and Gere (1961) and Ávila and Beck (2010).

The strong formulation of the stochastic problem for the Timoshenko beam, fixed the kth simple event is expressed by:

$$\begin{cases} \text{Find } w(x,\omega_k) \text{ e } \phi(x,\omega_k) \in L^2\big((\Omega,\mathcal{F},P)\otimes H^m(0,l)\big), \text{such that,} \\ \dfrac{d}{dx}\left[\left(EI\dfrac{d\phi}{dx}\right)(x,\omega_k)\right] + k_c AG\left(\dfrac{dw}{dx}-\phi\right)(x,\omega_k) = 0, \\ \dfrac{d}{dx}\left[k_c AG\left(\dfrac{dw}{dx}-\phi\right)(x,\omega_k)\right] = -q(x,\omega_k), \\ w(0,\omega) = w(l,\omega) = 0, \phi(0,\omega) = \phi(l,\omega) = 0, \forall x \in (0,l)\ e\ \omega \in (\Omega,F,P), \end{cases} \tag{P.6}$$

where $G(x,\omega)$ *is* the transverse modulus of elasticity, $A(x,\omega)$ is the cross-sectional area and k_c is the shear factor.

From the variational formulation, the elements of the linear system are identified for the kth realization of the stochastic process, such that the problem becomes:

$$\begin{cases} \text{To determine } w(x,\omega_k) \text{ e } \phi(x,\omega_k) \in L^2\big((\Omega,\mathcal{F},P)\otimes H^m(0,l)\big), \text{such that,} \\ \displaystyle\sum_{i=1}^{m}\left\{\int_0^l \left[K_c AG(x,\omega_k)\left(\frac{d\psi_i}{dx}.\psi_j\right)(x)\right]dx\right\} w_{ik} + \\ \displaystyle+\sum_{i=1}^{m}\left\{\int_0^l \left[EI(x,\omega_k)\left(\frac{d\psi_i}{dx}.\frac{d\psi_j}{dx}\right)(x) + K_c AG(x,\omega_k).\big(\psi_i.\psi_j\big)(x)\right]dx\right\}\phi_{ik} = 0 \\ \displaystyle\sum_{i=1}^{m}\left\{\int_0^l \left[EI(x,\omega_k)\left(\frac{d\psi_i}{dx}.\psi_j\right)(x)\right]dx\right\} w_{ik} + \\ \displaystyle-\sum_{i=1}^{m}\left\{\int_0^l \left[K_c AG(x,\omega_k).\big(\psi_i.\psi_j\big)(x)\right]dx\right\}\phi_{ik} = \int_0^l f(x,\omega_k).\psi_j(x)dx, \end{cases}, \tag{P.7}$$

where,

$$\begin{cases} a_k\big(\psi_i,\psi_j\big) = \displaystyle\int_0^l \left[EI(x,\omega_k)\left(\frac{d\psi_i}{dx}.\psi_j\right)(x)\right]dx, \\ b_k\big(\psi_i,\psi_j\big) = \displaystyle\int_0^l \left[K_c AG(x,\omega_k).\big(\psi_i.\psi_j\big)(x)\right]dx, \\ c_k\big(\psi_i,\psi_j\big) = -\displaystyle\int_0^l \left[K_c AG(x,\omega_k)\left(\frac{d\psi_i}{dx}.\psi_j\right)(x)\right]dx\,, \\ d_k\big(\psi_i,\psi_j\big) = \displaystyle\int_0^l \left[EI(x,\omega_k)\left(\frac{d\psi_i}{dx}.\frac{d\psi_j}{dx}\right)(x)\right]dx. \end{cases} \tag{22}$$

These coefficients form a system of linear equations, that is:

$$\begin{cases} a_k(\psi_i,\psi_j).w_k + b_k(\psi_i,\psi_j).\phi_k = F_k, \\ c_k(\psi_i,\psi_j).w_k + d_k(\psi_i,\psi_j).\phi_k = 0. \end{cases} \quad (23)$$

In the finite element method the stiffness matrix is decomposed into a portion associated with flexion $K_F = EI(x,\omega)$ and another portion associated with shear $K_C = k_c GA(x,\omega)$, thus $[K]^e = [K_F]^e + [K_C]^e$.

The flexibility matrix of the material or compliance $\left[S_{\omega_k}(x)\right]$ is the inverse of the stiffness matrix for one sample and the definitions of the symmetries for the flexibility are identical to those used for the stiffness, such that in the finite element method we have:

$$\left[S_{\omega_k}(x)\right]^e = \int_0^l \left\{ \begin{array}{l} EI_{\omega_k}(x)\begin{bmatrix} 0 & 0 \\ 0 & \left[\dfrac{dN_\phi}{dx}(x)\right]^t \left[\dfrac{dN_\phi}{dx}(x)\right] \end{bmatrix} + \\ K_c GA_{\omega_k}(x)\begin{bmatrix} \left[\dfrac{dN_w}{dx}(x)\right]^t \left[\dfrac{dN_w}{dx}(x)\right] & \left[\dfrac{dN_w}{dx}(x)\right]\left[N_\phi(x)\right] \\ \left[N_\phi(x)\right]^t \left[\dfrac{dN_w}{dx}(x)\right] & \left[N_\phi(x)\right]^t \left[N_\phi(x)\right] \end{bmatrix} \end{array} \right\} .J(x)dx. \quad (24)$$

where the Jacobian, $J = \sum_{i=1}^{n} N_{i,\xi} x_i$.

The integration solution uses the two-point Gauss-Legendre quadrature. The model is applied to linear elements, with linear interpolation functions for both displacements, being:

$$\begin{cases} N_1(x) = \dfrac{1-\psi(x)}{l}, \\ N_2(x) = \dfrac{\psi(x)}{l}. \end{cases} \quad (25)$$

So you can write for the element:

$$w_{\omega_k}(x) = \left(\frac{1-\psi(x)}{l}\right) w_1(x) + \left(\frac{\psi(x)}{l}\right) w_2(x), \quad (26)$$

$$\phi_{\omega_k}(x) = \left(\frac{1-\psi(x)}{l}\right) \phi_1(x) + \left(\frac{\psi(x)}{l}\right) \phi_2(x). \quad (27)$$

And, the stiffness matrix of the element,

$$K_{\omega_k}(x) = \begin{bmatrix} \frac{EI_{\omega_k}(x)}{l} + \frac{K_c GA_{\omega_k}(x).l}{3} & -\frac{EI_{\omega_k}(x)}{l} + \frac{K_c GA_{\omega_k}(x).l}{6} & -\frac{K_c GA_{\omega_k}(x)}{2} & \frac{K_c GA_{\omega_k}(x)}{2} \\ -\frac{EI_{\omega_k}(x)}{l} + \frac{K_c GA_{\omega_k}(x).l}{6} & \frac{EI_{\omega_k}(x)}{l} + \frac{K_c GA(x).l}{3} & -\frac{K_c GA_{\omega_k}(x)}{2} & \frac{K_c GA_{\omega_k}(x)}{2} \\ -\frac{K_c GA_{\omega_k}(x)}{2} & -\frac{K_c GA_{\omega_k}(x)}{2} & \frac{K_c GA_{\omega_k}(x)}{l} & -\frac{K_c GA_{\omega_k}(x)}{l} \\ \frac{K_c GA_{\omega_k}(x)}{2} & \frac{K_c GA_{\omega_k}(x)}{2} & -\frac{K_c GA_{\omega_k}(x)}{l} & \frac{K_c GA_{\omega_k}(x)}{l} \end{bmatrix}. \tag{28}$$

4. Numerical Results

This sextion aims to apply the methods discussed to calculate the effect of spatial and material variability on the solution of stochastic or uncertainty propagation systems in the Euler-Bernoulli and Timoshenko beams. The propagation of uncertainty is discussed for a bending problem of a beam of length $l = 1\ m$, fixed at both ends, transverse loading $f = 100x10^3\ Pa/m$. The random physical quantities are Young's modulus, $E = 400\ GPa$, with rectangular cross-section, $b = 1/30\ m$ e $h = 1/25\ m$, moment of inertia obtained by $I = bh^3/12$ and scalar rigidity $k = 3EI/L^3$. The uncertainty in these physical quantities is represented using parameterized stochastic processes. Mean values of k is $\mu_k = 71331\ Nm^2$, standard deviation $\sigma_k = 8198.6\ Nm^2$ and coefficient of variation, $\delta = 1/10\ \mu_k$.

For the formulation of the stochastic finite element method the results were obtained for 100 elements with 2 degrees of freedom per node and approximation functions given by the Hermite polynomials.

The relative error between the method λ-Neumann and the Monte Carlo simulation for vertical and angular displacements are obtained from:

$$E_{\%}(w_i) = \frac{|w_i - \mu_{w_i}|}{\mu_{w_i}} \tag{29}$$

$$E_{\%}(\phi_i) = \frac{|\phi_i - \mu_{\phi_i}|}{\mu_{\phi_i}} \tag{30}$$

where w_i is the vertical displacement, ϕ_i is the angular displacement obtained by the λ-Neumann method and μ_w, μ_ϕ the values obtained by the pure Monte Carlo simulation.

In the example, the uncertainty is assumed on the flexural stiffness with is modeled as a parameterized stochastic process:

$$k(x, \omega_k) = \mu_k + \sqrt{3}.\sigma_k \left[\xi_1(\omega)\cos\left(\frac{x}{l}\right) + \xi_2(\omega)\,\mathrm{sen}\left(\frac{x}{l}\right)\right] \tag{31}$$

4.1 – Euler Bernoulli beam

The convergence of the input parameters or the number simulations is appropriate for an accurate reference solution. The estimate of flexural stiffness expected value and variance are obtained from 100.000 sample of system response are illustrated in Fig. 1. Show that there was a significant decrease in the variation of the statistical moments. The uncertainty of flexural stiffness is modeled by a uniform distribution as a parameterized stochastic process with mean $\mu = 0$ and standard deviation, $\sigma = 1$.

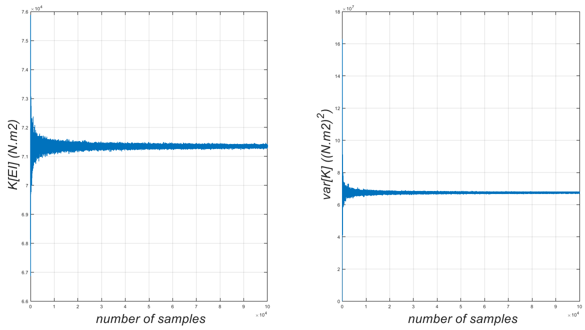

Fig. 1 Convergence of flexural stiffness

The uncertainty of flexural stiffness is modeled by a uniform distribution as a parameterized stochastic process with mean $\mu = 0$ and standard deviation, $\sigma = 1$, As shown in Fig. 2

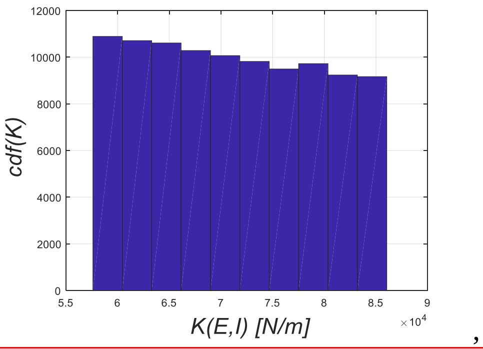

Fig. 2 Cumulative distribution function flexion to stiffness.

Fig. 3 shows the histogram of vertical and angular displacement.

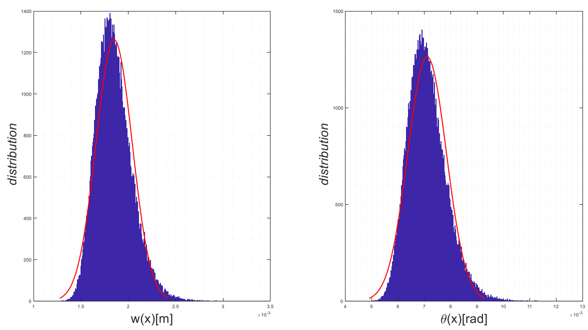

Fig. 3 Histogram of vertical (left) and angular (right) displacements.

Results on the uncertainties associated with the numerical and theoretical beam model are presented in Squarcio and Ávila (2017). Fig. 4 shows the variability of the displacements obtained by Monte Carlo simulation and by the λ-Neumann method.

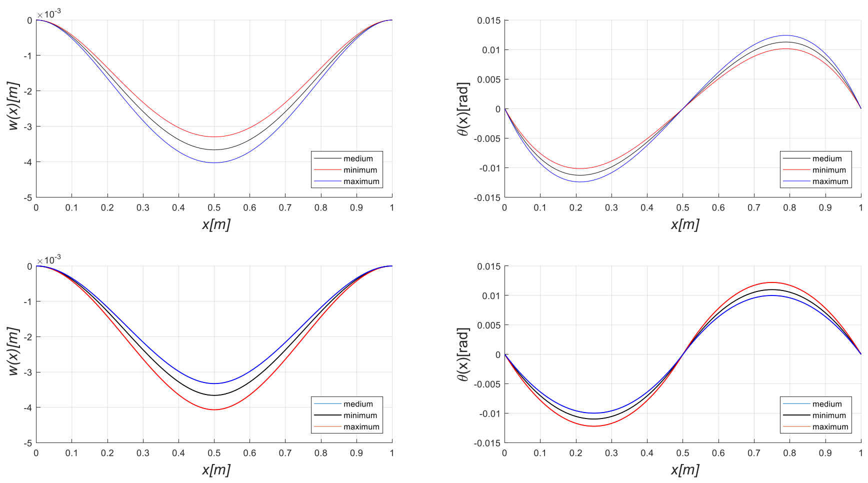

Fig. 4 Variability of displacements by Monte Carlo simulation (left) and the λ-Neumann method (right).

The difference between λ-Neumann method and pure Monte Carlo is in the order of 0. 12%. In both methods the variance of the vertical and angular displacement estimated was in the order of $1x10^{-6}$ N/m^2 for $x = 0.5m$ and $x = 0.25m$, respectively.

Temporal data obtained using the Neumann series are presented in Table 1.

Table 1 Processing Times (*seg*)

Monte Carlo	λ-Neumann
1.16612	0.12304

4.2 –Timoshenko beam

Considering the theoretical model of the Timoshenko beam, $15x10^3$ samples are used for the random process, as expressed in references. The finite element model, with results for 100 elements and 2 degrees of freedom per node, with quadratic Gaussian integration, $K_S = 5/6$ e $G = E/2(1+\nu)$.

The variation of Monte Carlo simulation (above) and λ-Neumann method (below) is shown in Fig. 5 for theoretical model. It is model considers the following equations to obtain the vertical displacement:

$$w_x = \frac{q}{24EI}(x^4 - 2lx^3 + l^2x^2) + \frac{q}{2K_cGA}(Lx - x^2). \qquad (32)$$

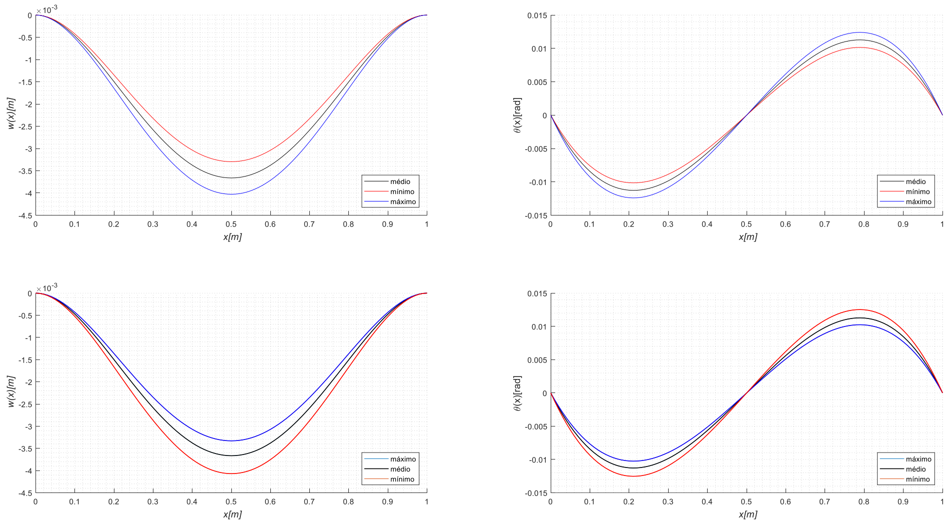

Fig. 5 Variation of vertical (left) and angular (right) displacements.

Table 4-5 shows the results for the coordinates $x = 0.25m$ and $x = 0.5m$ for both vertical and rotational displacements. Note that the uncertainty becomes more pronounced for the vertical displacement when the greater the proximity with respect to the center of gravity of the beam. In a different way the uncertainty behavior associated with the rotation is greater when the distance is *25%* of the total length of the beam.

Table 4 Variability of the displacements to the method Monte Carlo.

x(m)		0,25	0,50
w*	Minimum	-0.002166573723232	-0.004069788039730
	Mean	-0.001949883991981	-0.003662658051088
	Maximum	-0.001773028895480	-0.003330371823154
θ**	Minimum	-0.012349902008553	-6.507760863440181e-04
	Mean	-0.011113712647603	-5.856352881639037e-04
	Maximum	-0.010104775179909	-5.324694916963798e-04

Table 5 Variability of the displacements to the method *λ-Neumann*.

x(m)		0,25	0,50
w*	Minimum	-0.002166531467934	-0.004066453855960
	Mean	-0.001772996283555	-0.003327647276425
	Maximum	-0.001949478148220	-0.003658966985458
θ**	Minimum	-0.010104589132726	-6.507633836478925e-04
	Mean	-0.010104589132726	-5.324596879700113e-04
	Maximum	-0.011111397356810	-5.855132842906681e-04

*w: vertical displacement, *m*.

**θ: angular displacement, *rad*.

The relative error between the numerical and theoretical method for vertical and angular displacements is in the order of 0. 18%. In both methods the variance of the vertical and angular displacement estimated was in the order of $1x10^{-6}\ N/m^2$ for $x = 0.5m$ and $x = 0.25m$, respectively.

Table 6 Processing Times. (*seg*)

Monte Carlo	λ-Neumann
1.516498	0.225118

5. Conclusions

The present work presents a stochastic bending problem of Euller-Bernoulli and Timoshenko beam with uncertainty in the stiffness parameters. Stochastic finite element is used for obtaining approximate solutions, in selected examples, and the results obtained, in terms of statistic moments, were compared to reference values, estimated by Monte Carlo Simulations. Results of the application of the λ-Neumann method presented excellent agreement with the reference values of Monte Carlo simulation, in all examples.

In the stochastic model, it was observed that the propagation of uncertainty, present in the stiffness parameters of the beams, causes a deviation of the stochastic process of beam displacement in relation to the displacement obtained in the deterministic problem. The propagation of uncertainty proved to be more pronounced in examples in which the random parameters were related to higher order derivatives in the variational formulation. The example of uncertainty in the section height presented the greatest deviation from the expected value of the displacement stochastic process and also the highest coefficient of variation indicating a strong sensitivity in the mechanical response of the stochastic model, regarding the uncertain parameter.

In relation to the λ-Neumann method it is observed that the computational time depends on:

- The number of terms in the series.
- The number of elements and degrees of freedom in the system.
- The variability of entries.
- The level of accuracy required.

Further studies may consider the application of the proposed model to other linear elasticity problems (solid and plane finite elements) and the influence of the number of random variables used for describing parameterized stochastic processes, in the error estimates of the statistical moments, is a very interesting question to be investigated, as well as the application of other types of random variables in the description of parameters.

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