On Congruent Numbers

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Abstract

With respect to some classification of Pythagorean triples, if a number k is congruent then it can easily be proven. This expands the quest to resolve the congruent number problem. A proposition is put forward on rational sides forming a congruent number

Introduction

The congruent number problem is somehow a million-dollar question owing to the fact that Tunnel's Theorem leads to the Birch-Swinnerton Conjecture which Clay Mathematics has a cash prize of 1 million dollars in the Millennium Questions [1,2]

An integer number is congruent if it's equal to the area in a right-angled triangle of rational sides,[3].

This paper builds on a previous classification of Pythagorean triples[4]. If we indeed use the Archetypal equations, we can easily deduce faster methods of defining Congruent Numbers obtained by integer sides.

2 Statement of Results

2.1 Integer Sides

2.1.1 Archetype 1 Congruents

A brief description of Pythagorean triples a, b, c of Archetype 1 is that a :

a is an odd number greater than or equal to 3

$b = \frac{a^2-1}{2}$

$c = \frac{a^2+1}{2}$

The congruent number can be described as :

$$k = \frac{1}{2}(a \times b)$$

$$k = \frac{1}{2}(a \times \frac{a^2-1}{2})$$

$$k = \frac{a^3-a}{4}$$

Hence k is always a congruent number if a is an odd number.

k = 6,30,85,180,...

2.1.2 Archetype 2 & 3

A generalization of Archetype 2 and 3 for sides a, b, c is that :

$c = r^2 + z$

$b = r^2 - z$

$a = \sqrt{c^2 - b^2} = \sqrt{4zr^2} = 2r\hat{z}$

here $\hat{z} = \sqrt{z}$

Hence to solve for k:

$$k = \frac{1}{2}(a \times b$$

$$k = \frac{1}{2}(2r\hat{z} \times (r^2 - z))$$

$$k = \hat{z}(r^3 - rz) = \hat{z}(r^3 - r\hat{z}^2)$$

$$k = r^3\hat{z} - r\hat{z}^3$$

As long as $r^3\hat{z} > r\hat{z}^3$ we get nice solutions that form Pythagorean triples.

2.2 Congruent Number from Non-Integer Sides(atleast 1 non-integer side)

Here the main barrier is that all sides need to be rational. Taking this into account then:

$a = \frac{p_a}{q_a}$

$b = \frac{p_b}{q_b}$

$c = \frac{p_c}{q_c}$

If k is a congruent number then all p's and q's are integers

Using the integers p and q to represent fractions/rational numbers then we can conjecture several important statements from the behaviour.

Hypothesis 1

if $q_c = q_a q_b$ then (p_a, p_b, p_c) can be generated from archetypal equations.

Hypothesis 2

We can make a generalized version of Hypothesis 1 by letting :

$q_c = f(q_a, q_b)$, the numerators remain archetypal triples One such function f could an LCM. In such a case Hypothesis 1 holds if either q_a, q_b is 1 or they are prime to each other (no common factors).

An Analysis of David Golbergs' solutions for rational Pythagorean Triples having the Congruent Property obey Hypothesis 1 [5]

The area that requires rigorous effort is the values of $q_a q_b$ that completely factorizes a Pythagorean triple in the fashion

$$\frac{1}{q_a q_b}(A, B, C)$$

for A, B, C triple integers corresponding to $(p_a q_b, p_b q_a, p_c)$

From this we can dedice that :

i. C is uniquely determined

ii. A,B can never be prime numbers unless in a case where the denominator is 1.

iii. if k is a congruent number then :

$$k = \frac{1}{2q_a^2 q_b^2} AB$$

One major consequence is the following theorem.

Theorem *If (A, B, C) is an integer Pythagorean triple with A and B having at least one squared factor for each, then there must be a congruent number k with respect to the square factors.*

This Theorem combined with known divisibility methods means we can churn out congruent numbers very fast from non integer sides.

For Archetype 1 as with integers previously, we can right away pick out the congruent number.

In the regions 9, 25, 49,81.... we can calculate (A,B) sides

$(9, \frac{9^2-1}{2}) = (9, 8 \times 5)$

Here $k = 5$

$(25, \frac{25^2-1}{2}) = (25, 8 \times 39)$

Here $k = 39$

$(49, \frac{49^2-1}{2}) = (49, 16 \times 75)$

Here $k = 150$

A generalization would be that the congruent number depends on the prime factorization of B on the condition none has the integer 1 as a denominator. Otherwise, both A and B determine the congruence.

Archetypes 2 & 3

These ones require a combination of sides A and B and also usual square number distribution.

Some unique behavior

Consider the congruent number formed by the triple $(9, 40, 41)$ ie 180

After complete factorization $4 \times 5 \times 9$. From the rationals sides with at least 1 non-integer side we have 5, 45, as congruent numbers.

This behavior can also be extended to other triples.

Conclusion

Congruent numbers can described using archetypal equations

References

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