Kuhn-Tucker Theorem for Infinite Dimension

Manuel Alberto M. Ferreira

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MANUEL ALBERTO M. FERREIRA

Iscte-Instituto Universitário de Lisboa and ISTAR-IUL - Information Sciences, Technologies and Architecture Research Center Lisboa, Portugal

manuel.ferreira@iscte-iul.pt

ABSTRACT

This work objective is the presentation of Kuhn-Tucker theorem with the consideration of infinite dimension. So, the mathematical fundaments of this result, not so important in Mathematical Programming but a very challenging problem from the mathematical point of view, are shown in a very simple way. We will see how this result can be obtained in the context of real Hilbert spaces through the separation theorems.

Keywords: Hilbert spaces, separation theorems, Kuhn-Tucker theorem, infinite dimension.

1. INTRODUCTION

As an application of convex sets separation theorems, in real Hilbert spaces, see $[1-3.]$, the Kuhn-Tucker theorem for infinite dimension is presented. But consider first an important property of the real Hilbert spaces convex continuous functionals:

Theorem 1.1

A continuous convex functional in a Hilbert space has minimum in any limited closed convex set.

Demonstration:

If the space is of finite dimension, obviously the condition of the convexity for the set is not needed. In spaces of infinite dimension, note that if $\{x_n\}$ is a minimizing sequence, so, as the sequence is bounded, it is possible to work with a weakly convergent sequence and there is weak lower semi continuity, see for instance $[4]$: $\underline{\lim} f(x_n) \geq f(x)$, calling f (.) the functional, where *x* is the weak limit, and consequently the minimum is *f*(*x*). As a closed convex set is weakly closed, *x* belongs to the closed convex set. □

Now it is possible to establish a basic result characterizing the minimal point of a convex functional constrained by convex inequalities: the Kuhn-Tucker theorem, see for instance $[5]$, object of the next section. A finite number of inequalities will be considered, for now, and note that there is no need of imposing any continuity conditions, see $[1]$.

2. KUHN-TUCKER THEOREM

Let's begin with

Theorem 2.1 (Kuhn-Tucker)

Be $f(\boldsymbol{x}), f_i(\boldsymbol{x}), i = 1, \dots, n,$ convex functionals defined in a convex subset *C* of a Hilbert space. Consider the problem

$$\min_{x \in C} f(\boldsymbol{x})$$
$$sub: f_i(\boldsymbol{x}) \leq 0, i = 1, \dots, n.$$

Be $\boldsymbol{x}_0$ a point where the minimum, supposed finite, is reached. Suppose also that for each vector $\boldsymbol{u}$ in E_n (Euclidean space of dimension *n*), non-null and such that $u_k \geq 0$, there is a point $\boldsymbol{x}$ in *C* such that

$$\sum_{k=1}^{n} u_k f_k(\boldsymbol{x}) < 0 \qquad (2.1)$$

where u_k are the coordinates of $\boldsymbol{u}$.

Thus,

i) There is a vector $\boldsymbol{v}$, with non-negative coordinates v_k, such that

$$\min_{x \in C} \left\{ f(\boldsymbol{x}) + \sum_{k=1}^{n} v_k f_k(\boldsymbol{x}) \right\} = f(\boldsymbol{x}_0) + \sum_{k=1}^{n} v_k f_k(\boldsymbol{x}_0) = f(\boldsymbol{x}_0), \qquad (2.2)$$

ii) For any vector $\boldsymbol{u}$ in E_n with non-negative coordinates (it is also said: belonging to the positive cone of E_n)

$$f(\boldsymbol{x}) + \sum_{k=1}^{n} v_k f_k(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \sum_{k=1}^{n} v_k f_k\big(\boldsymbol{x}_0 \geq f(\boldsymbol{x}_0)\big) + \sum_{k=1}^{n} u_k f_k(\boldsymbol{x}_0). \qquad (2.3)$$

Demonstration:
Be the sets *A* and *B* in E_{n+1}:

$A: \{\boldsymbol{y} = (y_0, y_1, \dots, y_n) \in E_{n+1}: y_0 \geq f(\boldsymbol{x}), y_k \geq f_k(\boldsymbol{x})$ for some $\boldsymbol{x}$ in $C, k = 1, \dots, n.\}$,

$B: \{\boldsymbol{y} = (y_0, y_1, \dots, y_n) \in E_{n+1}: y_0 < f(x_0), y_i < 0, \qquad i = 1, \dots, n.\}$.

It is easy to confirm that *A* and *B* are disjoint convex sets in E_{n+1}.
So, they can be separated, that is, it is possible to find $v_k, k = 0,1, \dots, n$ such that

$$\inf_{x \in C} v_0 f(\boldsymbol{x}) + \sum_{k=1}^{n} f_k(\boldsymbol{x}) \geq v_0 f(x_0) - \sum_{k=1}^{n} v_k |y_k|. \qquad (2.4)$$

As (2.4) must hold for any $|y_k|$, it is concluded that $v_k, k = 1, \dots, n$, is non-negative. Approaching $|y_k|$ from zero it is obtained

$$v_0 f(x_0) + \sum_{k=1}^{n} v_k f_k(\boldsymbol{x}) \geq v_0 f(x_0)$$

and as the $f_k(x_0)$ are non-positive it follows that

$$\sum_{k=1}^{n} v_k f_k(x_0) = 0. \qquad (2.5)$$

Then it is shown that $\boldsymbol{v_0}$ must be positive

In fact if the whole $v_k, k = 1, \dots, n$ are zero, v_0 cannot be zero, and from $v_0 z_0 \geq v_0 y_0$ for any $y_0 < f(x_0) < z_0$, it follows that v_0 must be positive.

Supposing now that not all the v_k are zero, k=1, …, n, there is an $\boldsymbol{x} \in C$ such that $\sum_{k=1}^{n} v_k f_k(\boldsymbol{x}) < 0$ (by hypothesis). But for any z_0 greater or equal than $f(\boldsymbol{x})$ it must be $v_0\left(z_0 - f(x_0)\right) \geq -\sum_{k=1}^{n} v_k f_k(x_0) > 0$, and so v_0 must be positive. So, after (2.4) and putting $V_k = \frac{v_k}{v_0}, k = 1, \dots, n$ it is obtained

$$f(\boldsymbol{x}) + \sum_{k=1}^{n} V_k f_k(\boldsymbol{x}) \geq f(x_0) = f(x_0) + \sum_{k=1}^{n} V_k f_k(x_0),$$

resulting in consequence the remaining conclusions of the theorem. □

Observation:

- A sufficient condition, obvious but useful, so that (2.1) holds is that there is a point $\boldsymbol{x}$ in *C* such that $f_i(\boldsymbol{x})$ is lesser than zero for each $i, i = 1, \dots, n$.

Corollary 2.1 (Lagrange Duality Theorem)

In the conditions of Kuhn-Tucker's Theorem

$$f(x_0) = \sup_{\boldsymbol{u} \geq 0} \inf_{x \in C} \left(f(\boldsymbol{x}) + \sum_{k=1}^{n} u_k f_k(\boldsymbol{x}) \right).$$

Demonstration:

$\boldsymbol{u} \geq 0$ means that the whole coordinates$u_k, k = 1, \dots, n$, of $\boldsymbol{u}$ are non-negative. The result is a consequence of the arguments used in the Theorem of Kuhn-Tucker demonstration:

- For any $\boldsymbol{u} \geq 0$

$$\inf_{x \in C} \left(f(\boldsymbol{x}) + \sum_{k=1}^{n} u_k f_k(\boldsymbol{x}) \right) \leq f(x_0) + \sum_{k=1}^{n} u_k f_k(x_0) \leq f(x_0).$$

- For $u_k = v_k$

$$\inf_{x \in C}\left(f(\boldsymbol{x}) + \sum_{k=1}^{n} v_k\, f_k(\boldsymbol{x}) \right) \geq f(x_0).$$

then resulting the conclusion. □

Observation:

- This Corollary gives a process to determine the problem optimal solution.
- If the whole v_k in expression (2.3) are positive, x_0 is a point that belongs to the border of the convex set determined by the inequalities.
- If the whole v_k are zero, the inequalities are redundant for the problem, that is: the minimum is the same as in the "free" problem (without the inequalities restrictions).

3. KUHN-TUCKER THEOREM FOR INEQUALITIES IN INFINITE DIMENSION

In this section, the situation resulting from the consideration of infinite inequalities will be studied. A possible approach is:

- To consider a transformation *F*(*x*) from a real Hilbert space *H* to L_2: space of the summing square functions sequences.
- To consider the positive cone $\wp$, in L_2, of the sequences which the whole terms are non-negative.
- To consider the negative cone $\aleph$, in L_2, of the sequences which the whole terms are non-positive.
- To formalize the problem of the minimization of the convex functional *f*(*x*), constrained to $x \in C$ convex, as in section 2, and $F(x) \in \aleph$, supposing that $F(x)$ is convex.

Unfortunately, the Kuhn-Tucker theorem does not deal with this situation. So, similarly to the demonstration of Theorem 2.1 define

$$A = \{(y,z): y \geq f(x) \wedge z - F(x) \in \wp \text{ for any } x \in C\},$$

$$B = \{(y,z): y < f(x_0) \ \wedge \ z \in \aleph\},$$

where x_0 is a minimizing point, as before. But now, *A* and *B*, even being disjoint, can not necessarily be separated if neither *A* nor *B* have interior points. And evidently $\aleph$ has not interior points.

Another way, to establish a generalization, may be:

- To consider a real Hilbert space *I* that encloses a **closed convex cone** $\wp$.
- Given any two elements $x, y \in I$, $x \geq y$ if $x - y \in \wp$.
  It is a well-defined order relation: if $x \geq y$ and $y \geq z$, $x - y \in \wp$ and $y - z \in \wp$; being $\wp$ a convex cone, $(x-y)+(y-z) \in \wp$, that is $x \geq z$.

- So $\wp$ may be given by $\wp = \{x \in I: x \geq 0\}$ and may be called **positive cone**.
- The **negative cone** $\aleph$ will be given by $\aleph = -\wp = \{x \in I: x \leq 0\}$.

Having as reference this order relation, it is possible to define a convex transformation in the usual way. If the cone ℵ has a non-empty interior, a version of the **Kuhn-Tucker's theorem for infinite dimension** can be established.

Theorem 3.1 (Kuhn-Tucker in Infinite Dimension)

Call *C* a convex subset of a real Hilbert space *H* and $f(x)$ a real convex functional defined in *C*.

Be I a real Hilbert space with a convex closed cone $\wp$, with non-empty interior, and $F(x)$ a convex transformation from *H* to *I* – convex in relation with the order induced by the cone $\wp$.

Consider x_0, a minimizing of $f(x)$ in *C*, constrained to the inequality $F(x) \leq 0$.

Call $\wp^* = \{x: [x,p] \geq 0, \text{for any } p \in \wp\}$ - the dual cone.

Admit that given any $u \in \wp^*$ it is possible to determine x in C such that $[u, F(x)] < 0$.

So, there is an element v in the dual cone $\wp^*$, such that for x in C

$$f(x) + [v, F(x)] \geq f(x_0) + [v, F(x_0)] \geq f(x_0) + [u, F(x_0)],$$

where u is any element of $\wp^*$.

Demonstration:

It is identical to the one of Theorem 2.1. Build *A* and *B*, subsets of $E_1 \times I$:

$$A = \{(a,y): a \geq f(x), y \geq F(x), \quad \text{for any } x \text{ in } C\},$$
$$B = \{(a,y): a \leq f(x_0), y \leq 0\}.$$

In the real Hilbert space $E_1 \times I$, these sets can be separated, since *B* has non-empty interior and $A \cap B$ has not any interior point of *B*. So it is possible to find a number a_0 and $v \in I$ such that, for any x in *C*, $a_0 f(x) + [v, F(x)] \geq a_0 f(x_0) - [v, p]$ for any p in $\wp$. As this inequality left side is lesser than infinite, it follows that $[v, p] \geq 0$, for any $p \in \wp$ and so $v \in \wp^*$.

The remaining demonstration is a mere copy of the Theorem 2.1' s. □

There is also a version in infinite dimension for the Lagrange's Duality Theorem:

Corollary 3.1 (Lagrange's Duality Theorem in Infinite Dimension)

In the conditions of Kuhn-Tucker's Theorem in Infinite Dimension

$$f(x_0) = \sup_{v \in \wp^*} \inf_{x \in C} (f(x) + [v, F(x)]).$$

4. CONCLUSIONS

Through subtle, although conceptually complicated, generalization of Kuhn-Tucker's theorem it was possible to present the mathematical fundaments of Kuhn-Tucker's theorem in infinite dimension. It was necessary to define very carefully the domains to be considered: the Hilbert spaces and the adequate cones. And this is a really challenging problem from the mathematical point of view.

To attain such an achievement, it was necessary to use a lot of mathematical tools that may be considered in the scope of the functional analysis. So, as in [1], in Kolmogorov and Fomin [4] the chapters used were mainly III and IV; in Balakrishnan [5] 1 and 2; in Kantorovich and Akilov [6] II and IV; in Brézis [7] I and V; in Royden [8] 10; in Aubin [9] 1, 2, 3 and 4. References [10 − 18] constitute a short collection of works on this subject and related ones.

REFERENCES

1. M. A. M. Ferreira, M. Andrade and M. C. P. Matos (2010). Separation Theorems in Hilbert Spaces Convex Programming. Journal of Mathematics and Technology, 1 (5), 20-27.

2. M. A. M. Ferreira and M. Andrade (2011). Hahn-Banach Theorem for Normed Spaces. International Journal of Academic Research, 3 (4, I Part), 13-16.

3. M. A. M. Ferreira and M. Andrade (2011). Riesz Representation theorem in Hilbert Spaces Separation Theorems. International Journal of Academic Research, 3 (6, II Part), 302-304.

4. A. N. Kolmogorov and S. V. Fomin (1982). Elementos da Teoria das Funções e de Análise Funcional, Editora Mir.

5. A. V. Balakrishnan (1981). Applied Functional Analysis, Springer-Verlag New York Inc., New York.

6. L. V. Kantorovich and G. P. Akilov (1982). Functional Analysis, Pergamon Press, Oxford.

7. H. Brézis (1983). Analyse Fonctionelle (Théorie et Applications), Masson, Paris.

8. H. L. Royden (1968). Real Analysis, Mac Millan Publishing Co. Inc., New York.

9. J. P. Aubin (1979). Applied Functional Analysis, John Wiley & Sons Inc., New York.

10. J. von Neumann and O. Morgenstern (1967). Theory of Games and Economic Behavior, John Wiley & Sons Inc., New York.

11. S. Kakutani (1941). A Generalization of Brouwer's Fixed Point Theorem, Duke Mathematics Journal, 8.

12. J. Nash (1951). Non-Cooperative Games, Annals of Mathematics, 54.

13. M. A. M. Ferreira (1986). Aplicação dos Teoremas de Separação na Programação Convexa em Espaços de Hilbert, Revista de Gestão, I (2), 41-44.

14. M. A. M. Ferreira and M. Andrade (2011). Management Optimization Problems. International Journal of Academic Research, 3 (2, Part III), 647-654.

15. M. C. Matos and M. A. M. Ferreira (2006). Game Representation -Code Form. Namatame, Akira (ed.) et al., The complex networks of economic interactions. Essays in agent-based economics and econophysics. Selected papers based on the presentation at the 9th international workshop on heterogeneous interacting agents (WEHIA), Kyoto, Japan, May 27–29, 2004. Berlin: Springer (ISBN 3-540-28726-4/pbk). Lecture Notes in Economics and Mathematical Systems 567, 321-334. 10.1007/3-540-28727-2_22

16. M. A. M. Ferreira (2020). Some considerations on orthogonality, strict separation theorems and applications in Hilbert spaces. In Le Bin Ho (Ed.). Hilbert spaces: Properties and applications (pp.1-19). Nova Science Publishers. http://hdl.handle.net/10071/26471

17. M. A. M. Ferreira (2020). Convex programming based on Hahn-Banach theorem. In Luigi Giacomo Rodino (Ed.). Theory and Applications of Mathematical Science Vol. 1 (pp. 60-72). Book Publisher International. http://doi.org/10.9734/bpi/tams/v1

18. M. Bachir, A. Fabre & S. Tapia-García (2021). Finitely determined functions. *Adv. Oper. Theory* **6**, 28.

https://doi.org/10.1007/s43036-020-00125-y