



Properties of the First Possible Counterexample in the Robin's Inequality

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ABSTRACT. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. In 1915, Ramanujan proved that under the assumption of the Riemann Hypothesis, the inequality $\sigma(n) < e^\gamma \times n \times \ln \ln n$ holds for all sufficiently large n , where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann Hypothesis is true. This inequality is known today as the Robin's inequality. We demonstrate an interesting result about the smallest possible counterexample exceeding 5040 of the Robin's inequality. The existence of such counterexample seems unlikely according to the evidence of this result. In this way, we provide a new step forward in the efforts of trying to prove the Riemann Hypothesis.

1. INTRODUCTION

$\sigma(n)$ is the sum-of-divisors function of n [1]:

$$\sum_{d|n} d.$$

Define $s(n)$ to be $\frac{\sigma(n)}{n}$. Say the Robin's inequality is satisfied for n when

$$s(n) < e^\gamma \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and \log is the natural logarithm. The importance of this property is: if the Robin's inequality is satisfied for all $n > 5040$, then the Riemann Hypothesis is true [3]. There are several known results about the possible counterexamples of the Robin's inequality exceeding 5040 [1]. This our main Theorem:

Theorem 1.1. [\[main\]](#) *Let $n > 5040$ and $n = r \times q$, where q denotes the largest prime factor of n . If $n > 5040$ is the smallest integer such that n does not satisfy the Robin's inequality, then*

$$1.0000000001 \times \log \log r > \log \log n.$$

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2. A CENTRAL LEMMA

Lemma 2.1. [central] *Suppose that $n > 5040$ and let $n = r \times q$, where q denotes the largest prime factor of n . We have*

$$s(n) \leq \left(1 + \frac{1}{q}\right) \times s(r).$$

Proof. Suppose that n is the form of $m \times q^k$ where $q \nmid m$ and m and k are natural numbers. We have that

$$s(n) = s(m \times q^k) = s(m) \times s(q^k)$$

since s is multiplicative and m and q are coprimes [4]. However, we note that

$$s(q^k) \leq s(q^{k-1}) \times s(q)$$

due to we know that $s(a \times b) \leq s(a) \times s(b)$ when $a, b \geq 2$ [4]. In this way, we obtain that

$$s(q^{k-1}) \times s(q) = s(q^{k-1}) \times \left(1 + \frac{1}{q}\right)$$

according to the value of $s(q)$ [4]. In addition, we analyze that

$$s(m) \times s(q^{k-1}) = s(m \times q^{k-1}) = s(r)$$

because s is multiplicative and m and q are coprimes [4]. Finally, we obtain that

$$s(n) = s(m) \times s(q^k) \leq s(m) \times s(q^{k-1}) \times s(q) = s(r) \times \left(1 + \frac{1}{q}\right)$$

and as a consequence, the proof is finished. \square

3. PROOF OF MAIN THEOREM

Proof. Suppose that n is the smallest integer exceeding 5040 that does not satisfy the Robin's inequality. Let $n = r \times q$, where q denotes the largest prime factor of n . Let's also assume that $\frac{\log \log r}{\log \log n} \leq 0.9999999999$. In this way, the following inequality

$$s(n) \geq e^\gamma \times \log \log n$$

should be true. We know that

$$\left(1 + \frac{1}{q}\right) \times s(r) \geq s(q \times r) \geq s(n) \geq e^\gamma \times \log \log n$$

due to Lemma 2.1 [central]. Besides, this shows that

$$\left(1 + \frac{1}{q}\right) \times e^\gamma \times \log \log r > e^\gamma \times \log \log n$$

should be true as well. Certainly, if n is the smallest counterexample of the Robin's inequality exceeding 5040, then the Robin's inequality is satisfied on r [1]. That is the same as

$$\left(1 + \frac{1}{q}\right) \times \log \log r > \log \log n.$$

We have that

$$\left(1 + \frac{1}{q}\right) \times \log \log r > \log(\log r + \log q)$$

where we notice that $\log(a + c) = \log a + \log\left(1 + \frac{c}{a}\right)$. This follows

$$\left(1 + \frac{1}{q}\right) \times \log \log r > \log \log r + \log\left(1 + \frac{\log q}{\log r}\right)$$

which is equal to

$$(1 + q) \times \log \log r > q \times \log \log r + q \times \log\left(1 + \frac{\log q}{\log r}\right)$$

and thus,

$$\log \log r > q \times \log\left(1 + \frac{\log q}{\log r}\right).$$

This implies that

$$\begin{aligned} \frac{\log \log r}{\log\left(1 + \frac{\log q}{\log r}\right)} &= \\ \frac{\log \log r}{\log \frac{\log r + \log q}{\log r}} &= \\ \frac{\log \log r}{\log \frac{\log n}{\log r}} &= \\ \frac{\log \log r}{\log \log n - \log \log r} &= \\ \frac{\log \log r}{\log \log n \times \left(1 - \frac{\log \log r}{\log \log n}\right)} &= \\ \frac{\frac{\log \log r}{\log \log n}}{\left(1 - \frac{\log \log r}{\log \log n}\right)} &> q \end{aligned}$$

should be true. We know the function $\frac{x}{1-x}$ is monotone increasing and continuous in the interval $0 \leq x < 1$ and as a consequence,

$$\frac{\frac{\log \log r}{\log \log n}}{\left(1 - \frac{\log \log r}{\log \log n}\right)} \leq \frac{0.9999999999}{(1 - 0.9999999999)} = 9999999171.59636$$

when $\frac{\log \log r}{\log \log n} \leq 0.9999999999$. From the article reference [4], we know that

$$q > (\log n) \times e^{-\frac{1}{\log q}}.$$

Consequently, we obtain that

$$e^{\frac{1}{\log q}} \times 9999999171.59636 > \log n$$

which can be transformed into

$$2 \times 9999999171.59636 > \log n$$

that is equivalent to

$$e^{2 \times 9999999171.59636} > n.$$

We know that every possible counterexample n of the Robin's inequality exceeding 5040 must comply with $n > 10^{10^{10}}$ [2]. In conclusion, we obtain that

$$e^{2 \times 9999999171.59636} > n > 10^{10^{10}}$$

which is a trivial contradiction and therefore, it is not possible that $\frac{\log \log r}{\log \log n} \leq 0.9999999999$ when n is the smallest counterexample of the Robin's inequality exceeding 5040. In this way, we have

$$\frac{\log \log r}{\log \log n} > 0.9999999999$$

and

$$1.0000000001 \times \log \log r > \log \log n$$

and thus, the proof is completed. \square

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