Local Half-Derivation on of N-Dimensional Naturally Graded Quasi-Filiform Leibniz Algebra of Type II

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Abstract. In the present paper $\frac{1}{2}$-derivations and local $\frac{1}{2}$-derivations of n -dimensional naturally graded quasi-filiform Leibniz algebra of type II are studied. Namely, a common form of the matrix of $\frac{1}{2}$-derivations and local $\frac{1}{2}$-derivations of these algebras is clarified. It turns out that the common form of the matrix of a $\frac{1}{2}$-derivations on these algebras does not coincide with the local$\frac{1}{2}$-derivations' matrices common form on these algebras. Therefore, these n -dimensional naturally graded quasi-filiform Leibniz algebra of type II have local $\frac{1}{2}$-derivations that are not $\frac{1}{2}$-derivations.

INTRODUCTION

Filippov studied $\boldsymbol{\delta}$-derivations of Lie algebras in a series of papers [12-14]. The space of $\boldsymbol{\delta}$-derivations includes usual derivations ($\boldsymbol{\delta = 1}$), anti-derivations ($\boldsymbol{\delta = -1}$) and elements from the centroid. In [13] it was proved that prime Lie algebras, as a rule, do not have nonzero $\boldsymbol{\delta}$-derivations (provided $\boldsymbol{\delta \neq 1, -1, 0, \frac{1}{2}}$), and all $\boldsymbol{\frac{1}{2}}$-derivations of an arbitrary prime Lie algebra $\boldsymbol{A}$ over the field $\mathbb{F}$ of characteristic $\boldsymbol{p \neq 2, 3}$ with a non-degenerate symmetric invariant bilinear form were described. It was proved that if $\boldsymbol{A}$ is a central simple Lie algebra over a field of characteristic $\boldsymbol{p \neq 2, 3}$ with a nondegenerate symmetric invariant bilinear form, then any $\boldsymbol{\frac{1}{2}}$-derivation $\boldsymbol{\varphi}$ has the form $\boldsymbol{\varphi = \lambda x}$ for some $\boldsymbol{\lambda} \in \mathbb{F}$.

In recent decades, a well-known and active direction in the study of derivations of associative algebras and rings is the problem about local derivations. The notion of local derivation on algebras was introduced by R.V. Kadison [15], D.R. Larson and A.R. Sourour [18]. A local derivation on an algebra A is a linear map $\Delta: A \to A$ which satisfies that for any $x \in A$, there exists a derivation $D_x: A \to A$
(depending on x) such that $\Delta(x) = D_x(x)$.The main problems concerning local derivations are to find conditions under which local derivations become
derivations and to present examples of algebras with local derivations that are not derivations. Several authors investigated local derivations for finite or infinite dimensional Lie algebras [1-9, 11, 16, 17, 19, 20], and it was proved that every local derivation on many Lie algebras (for examples, semi-simple Lie algebras, Borel subalgebras of finite-dimensional simple Lie algebras, the Schrödinger algebra s_n in (n + 1)-dimensional space-time) is a derivation.

Investigation of local and 2-local δ-derivations on Lie algebras was initiated in [17] by A. Khudoyberdiyev and B. Yusupov. Namely, in [17] it is proved we introduce the notion of local and 2-local δ-derivations and describe local and 2-local $\frac{1}{2}$-derivation of finite-dimensional solvable Lie algebras with filiform, Heisenberg, abelian nilradicals. Moreover, we give the description of local $\frac{1}{2}$-derivation of oscillator Lie algebras, conformal perfect Lie algebras, and Schrödinger algebras. B. Yusupov, V. Vaisova and T. Madrakhimov proved similar results concerning local $\frac{1}{2}$-derivations of naturally graded quasi-filiform Leibniz algebras of type I in their recent paper [20]. They proved that quasi-filiform Leibniz algebras of type I, as a rule, admit local $\frac{1}{2}$-derivations which are not $\frac{1}{2}$-derivations.

PRELIMINARIES

In this section we recall some basic notions and concepts used throughout the paper.

For a given Leibniz algebra $(L, [\cdot,\cdot])$, the sequence of two-sided ideals are defined recursively as follows:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1.$$

This sequence is said to be the lower central series of L.

Definition 2.1 *A Leibniz algebra L is said to be nilpotent, if there exists $n \in \mathbb{N}$ such that $L^n = \{0\}$.*

Now we give the definitions of $\frac{1}{2}$-derivation and local $\frac{1}{2}$-derivation.

Definition 2.2 *Let $(L, [-,-])$ be an algebra with a multiplication $[-,-]$. A linear map D is called a δ-derivation if it satisfies*

$$D[x, y] = \delta([D(x), y] + [x, D(y)])$$

where δ from the ground field $\mathbb{F}$.

Definition 2.3 *A linear map Δ is called a local δ-derivation, if for any $x \in L$ there exists a δ-derivation $D_x: L \to L$ (depending on x) such that $\Delta(x) = D_x(x)$. The set of all local δ-derivations on L we denote by $LocDer_\delta(L)$.*

In this work, we focus on investigating local and 2-local $\frac{1}{2}$-derivations. Since any $\frac{1}{2}$-derivation is a local and 2-local $\frac{1}{2}$-derivations, we are interesting on local and 2-

local $\frac{1}{2}$-derivation, which is not a $\frac{1}{2}$-derivation. Such local (resp. 2-local) $\frac{1}{2}$-derivations we call non-trivial local (resp. 2-local) $\frac{1}{2}$-derivations.

Below we define the notion of a quasi-filiform Leibniz algebra.

Definition 2.4 *A Leibniz algebra L is called quasi-filiform if $L^{n-2} \neq \{0\}$ and $L^{n-1} = \{0\}$, where $n = dimL$.* Given an n-dimensional nilpotent Leibniz algebra L such that $L^{s-1} \neq \{0\}$ and $L^s = \{0\}$, put $L_i = L^i/L^{i+1},\ 1 \leq i \leq s-1$, and $gr(L) = L_1 \oplus \cdots \oplus L_{s-1}$. Due to $[L_i, L_j] \subseteq L_{i+j}$ we obtain the graded algebra $gr(L)$. If $gr(L)$ and L are isomorphic, $gr(L) \cong L$, we say that L is *naturally graded*.

Let x be a nilpotent element of the set $L \backslash L^2$. For the nilpotent operator of right multiplication $\mathcal{R}_x$ we define a decreasing sequence $C(x) = (n_1, n_2, \dots, n_k)$, where $n = n_1 + n_2 + \cdots + n_k$, which consists of the dimensions of Jordan blocks of the operator $\mathcal{R}_x$. On the set of such sequences we consider the lexicographic order, that is, $C(x) = (n_1, n_2, \dots, n_k) \leq C(y) = (m_1, m_2, \dots, m_t)$ iff there exists $i \in \mathbb{N}$ such that $n_j = m_j$ for any $j < i$ and $n_i < m_i$.

Definition 2.5 *The sequence $C(L) = \max\limits_{x \in L \backslash L^2} C(x)$ is called the characteristic sequence of the algebra L.*

Definition 2.6 *A quasi-filiform non Lie Leibniz algebra L is called an algebra of the type I (respectively, type II) if there exists an element $x \in L \backslash L^2$ such that the operator $\mathcal{R}_x$ has the form $\begin{pmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{pmatrix}$ (respectively, $\begin{pmatrix} J_2 & 0 \\ 0 & J_{n-2} \end{pmatrix}$).*

The following theorem obtained in [10] gives the classification of naturally graded quasi-filiform Leibniz algebras.

Theorem 2.1 [10] *An arbitrary n-dimensional naturally graded quasi-filiform Leibniz algebra of type II is isomorphic to one of the following pairwise non-isomorphic algebras of the families:neven*

$$\mathcal{L}_n^1: \begin{cases} [e_1, e_1] = e_2, \\ [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [e_1, e_i] = -e_{i+1}, & 3 \leq i \leq n-1, \end{cases}$$

$$\mathcal{L}_n^2: \begin{cases} [e_1, e_1] = e_2, \\ [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [e_1, e_3] = e_2 - e_4, \\ [e_1, e_i] = -e_{i+1}, & 4 \leq i \leq n-1, \end{cases}$$

$$\mathcal{L}_n^3: \begin{cases} [e_1, e_1] = e_2, \\ [e_i, e_1] = e_{i+1},\ 3 \leq i \leq n-1, \\ [e_1, e_i] = -e_{i+1},\ 3 \leq i \leq n-1, \\ [e_3, e_3] = e_2, \end{cases}$$

$$\mathcal{L}_n^4: \begin{cases} [e_1, e_1] = e_1, \\ [e_i, e_1] = e_{i+1},\ 3 \leq i \leq n-1, \\ [e_1, e_3] = 2e_2 - e_4, \\ [e_1, e_i] = -e_{i+1},\ 4 \leq i \leq n-1, \\ [e_3, e_3] = e_2, \end{cases}$$

n odd, $\mathcal{L}_n^1, \mathcal{L}_n^2, \mathcal{L}_n^3, \mathcal{L}_n^4$,

$$\mathcal{L}_n^5: \begin{cases} [e_1, e_1] = e_2, e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [e_1, e_i] = -e_{i+1}, & 3 \leq i \leq n-1, \\ [e_i, e_{n+2-i}] = (-1)^i e_n, & 3 \leq i \leq n-1, \end{cases}$$

$$\mathcal{L}_n^{6,\beta}: \begin{cases} [e_1, e_1] = e_2, \\ [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1 \\ [e_1, e_3] = \beta e_2 - e_4, & \beta \in \{1,2\} \\ [e_1, e_i] = -e_{i+1}, & 4 \leq i \leq n-1, \\ [e_i, e_{n+2-i}] = (-1)^i e_n, & 3 \leq i \leq n-1, \end{cases}$$

$$\mathcal{L}_n^{7,\gamma}: \begin{cases} [e_1, e_1] = e_2, \\ [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [e_1, e_i] = -e_{i+1}, & 3 \leq i \leq n-1, \\ [e_3, e_3] = \gamma e_2, & \gamma \neq 0, \\ [e_i, e_{n+2-i}] = (-1)^i e_n, & 3 \leq i \leq n-1, \end{cases}$$

$$\mathcal{L}_n^{8,\beta,\gamma}: \begin{cases} [e_1, e_1] = e_2, \\ [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [e_1, e_3] = \beta e_2 - e_4, \\ [e_1, e_i] = -e_{i+1}, & 4 \leq i \leq n-1, \\ [e_3, e_3] = \gamma e_2, & (\beta, \gamma) = (-2,1), (2,1)\ or (4,2), \\ [e_i, e_{n+2-i}] = (-1)^i e_n, & 3 \leq i \leq n-1, \end{cases}$$

where $\{e_1, e_2, \dots, e_n\}$ is a basis of the algebra.

MAIN RESULTS

Now let us give the main result about $\frac{1}{2}$-derivation and local $\frac{1}{2}$-derivation of naturally graded quasi-filiform Leibniz algebra of type II.

Proposition 3.1 *Any $\frac{1}{2}$-derivation of naturally graded quasi-filiform Leibniz algebra of type II has the following form:*

For algebras $\mathcal{L}_n^1$:

$$\begin{cases} D(e_1) = \sum_{i=1}^{n} a_i e_i \quad D(e_2) = a_1 e_2, D(e_3) = \sum_{i=1}^{n} b_i e_i\,, \\ D(e_j) = (1 - 2^{3-j}) a_1 e_j + 2^{3-j} \sum_{i=j}^{n} b_{i-j+3} e_i\ , \end{cases}$$

for algebras $\mathcal{L}_n^2$:

$$\begin{cases} D(e_1) = \sum_{i=1}^{n} \alpha_i e_i\,, \quad D(e_2) = \frac{1}{2}(2\alpha_1 + \alpha_3) e_2, \\ D(e_3) = \sum_{i=2}^{n} b_i e_i, \\ D(e_j) = (1 - 2^{3-j}) \alpha_1 e_j + 2^{3-j} \sum_{i=j}^{n} b_{i-j+3} e_i 4 \leq j \leq n \end{cases}$$

for algebras $\mathcal{L}_n^3$:

$$\begin{cases} D(e_1) = \sum_{i=1}^{n} \alpha_i e_i\,, \quad D(e_2) = \alpha_1 e_2, \quad D(e_3) = \sum_{i=2}^{n} b_i e_i\,, \\ D(e_j) = (1 - 2^{3-j}) \alpha_1 e_j + 2^{3-j} \sum_{i=j}^{n} b_{i-j+3} e_i, 4 \leq j \leq n. \end{cases}$$

for algebras $\mathcal{L}_n^4$:

$$\begin{cases} D(e_1) = \sum_{i=1}^{n} \alpha_i e_i\,, D(e_2) = (\alpha_1 + \alpha_3) e_2, D(e_3) = \sum_{i=2}^{n} b_i e_i\,, \\ D(e_j) = (1 - 2^{3-j}) \alpha_1 e_j + 2^{3-j} \sum_{i=j}^{n} b_{i-j+3} e_i, 4 \leq j \leq n, \end{cases}$$

where $b_3 = \alpha_1 + \alpha_3$.

for algebras $\mathcal{L}_n^5$:

$$\begin{cases} D(e_1)=\sum_{i=1}^{n}\alpha_i e_i, \quad D(e_2)=\alpha_1 e_2, \\ D(e_3)=\sum_{i=2}^{n-3} b_i\cdot e_i + b_{n-1}e_{n-1}+b_n e_n, \\ D(e_j)=2^{3-j}\left(\sum_{i=j}^{n} b_{i-j+3}e_i\right)+ \\ +(1-2^{3-j})\alpha_1 e_j+\dfrac{(-1)^{j-1}\alpha_{n+3-j}e_n}{2}, 4\le j\le n. \end{cases}$$

for algebras $\mathcal{L}_n^{6,\beta}(\beta=\{1;2\})$:

$$\begin{cases} D(e_1)=\sum_{i=1}^{n-2}\alpha_i e_i+\alpha_n e_n,, \quad D(e_2)=\frac{1}{2}\beta\alpha_3 e_2 \\ D(e_3)=\sum_{i=2}^{n-3} b_i\cdot e_i + b_{n-1}e_{n-1}+b_n e_n, \\ D(e_j)=2^{3-j}\left(\sum_{i=j}^{n} b_{i-j+3}e_i\right)+(1-2^{3-j})\alpha_1 e_j+ \\ \quad +\dfrac{(-1)^{j-1}\alpha_{n+3-j}e_n}{2}, 4\le j\le n \end{cases}$$

for algebras $\mathcal{L}_n^{7,\gamma}(\gamma\neq 0)$:

$$\begin{cases} D(e_1)=\sum_{i=1}^{n-2}\alpha_i e_i+\alpha_n e_n, \quad D(e_2)=\alpha_1 e_2, \\ D(e_3)=\sum_{i=2}^{n-3} b_i\cdot e_i + b_{n-1}e_{n-1}+b_n e_n, \\ D(e_4)=\frac{1}{2}\left(b_1 e_2+\alpha_1 e_4+\alpha_3\gamma e_2-\alpha_{n-1}e_n+\sum_{i=4}^{n} b_{i-1}e_i\right), \\ D(e_j)=2^{3-j}\left(\sum_{i=j}^{n} b_{i-j+3}e_i\right)+(1-2^{3-j})\alpha_1 e_j \\ \quad +\dfrac{(-1)^{j-1}\alpha_{n+3-j}e_n}{2}, 5\le j\le n \end{cases}$$

where $\alpha_1=\gamma b_3$

for algebras $\mathcal{L}_n^{8,\beta,\gamma}$:

$$\begin{cases} D(e_1)=\sum_{i=1}^{n}\alpha_i e_i, \quad D(e_2)=\frac{1}{2}(2\alpha_1+\alpha_3\beta)e_2, \\ D(e_3)=\sum_{i=2}^{n-3} b_i\cdot e_i + b_{n-1}e_{n-1}+b_n e_n, \\ D(e_4)=\frac{1}{2}\left(b_1 e_2+\sum_{i=4}^{n} b_{i-1}e_i+\alpha_1 e_4+\alpha_3\gamma e_2-b_{n-1}e_n\right), \\ D(e_j)=2^{3-j}\left(\sum_{i=j}^{n} b_{i-j+3}e_i\right)+(1-2^{3-j})\alpha_1 e_j \\ +\dfrac{(-1)^{j-1}\alpha_{n+3-j}e_n}{2}, 5\le j\le n \end{cases}$$

where $(\beta,\gamma)=(-2;1),(2;1)v\alpha(4;2)$ and $2\gamma\alpha_1+\gamma\alpha_3\beta=b_1\beta+2\gamma b_3$.

Proof. We prove the proposition for the algebra $\mathcal{L}_n^1$. The rest of the cases can be proved similarly. Let D be a $\frac{1}{2}$-derivation of the algebra. We set

$$D(e_1)=\sum_{i=1}^{n} a_i e_i, \quad D(e_3)=\sum_{i=1}^{n} b_i e_i.$$

Consider the equalities

$$D(e_2)=D([e_i\ e_i])=\frac{1}{2}\cdot([d(e_1),e_1]+[e_1,d(e_1)])=$$

$$=\frac{1}{2}\cdot\left(\left[\sum_{i=1}^{n}\alpha_i e_i, e_1\right]+\left[e_1,\sum_{i=1}^{n}\alpha_i e_i\right]\right)=\alpha_1 e_2.$$

Consider the equalities

$$D(e_4)=D([e_3,e_1])=\tfrac{1}{2}([D(e_3),e_1]+[e_3,D(e_1)])$$
$$=\tfrac{1}{2}([\textstyle\sum_{i=1}^{n} b_i e_i, e_1]+[e_3,\sum_{i=1}^{n} a_i e_i])=$$
$$=\tfrac{1}{2}(b_1 e_2+\textstyle\sum_{i=4}^{n} b_{i-1}e_i+a_1 e_4)=$$
$$=\frac{1}{2}b_1 e_2+\frac{1}{2}(a_1+b_3)e_4+\frac{1}{2}\sum_{i=5}^{n} b_{i-1}e_i$$

From the equalit

$$0=D([e_4,e_3])=\frac{1}{2}([D(e_4),e_3]+[e_4,D(e_3)])=$$
$$=\tfrac{1}{2}\left[\tfrac{1}{2}b_1 e_2+\tfrac{1}{2}(a_1+b_3)e_4+\tfrac{1}{2}\textstyle\sum_{i=5}^{n} b_{i-1}e_i, e_3\right]+$$
$$+\tfrac{1}{2}[e_4,\textstyle\sum_{i=1}^{n} b_i e_i]=b_1 e_5$$

which implies $b_1=0$.

With similar arguments applied on the products $[e_i,e_1]=e_{i+1}$ and with an induction on i, it is easy to check that the following identities hold for $5\le i\le n$:

$$D(e_i)=\left(1-\left(\frac{1}{2}\right)^{i-3}\right)a_1 e_i+\frac{1}{2^{i-3}}\sum_{j=i}^{n} b_{j-i+3}e_j.$$

Theorem 3.2 *Any local $\frac{1}{2}$-derivation of naturally graded quasi-filiform Leibniz algebra of type II has the following form:*

for the algebras $\mathcal{L}_n^1,\ \mathcal{L}_n^2$:

$$\mathrm{LocDer}_{\frac{1}{2}}:\begin{cases}\Delta(e_1)=\sum_{j=1}^{n} c_{j,1}e_j,\ \Delta(e_2)=c_{2,2}e_2, \\ \Delta(e_3)=\sum_{j=2}^{n} c_{j,3}e_j, \Delta(e_i)=\sum_{j=i}^{n} c_{j,i}e_i,\end{cases}$$

for the algebras $\mathcal{L}_n^3,\ \mathcal{L}_n^4$:

$$\mathrm{LocDer}_{\frac{1}{2}}:\begin{cases}\Delta(e_1)=\sum_{j=1}^{n} c_{j,1}e_j, \quad \Delta(e_2)=c_{2,2}e_2, \\ \Delta(e_3)=\sum_{j=2}^{n} c_{j,3}e_j, \Delta(e_4)=c_{2,4}e_2+\sum_{j=4}^{n} c_{j,4}e_j, \\ \Delta(e_i)=\sum_{j=i}^{n} c_{j,i}e_i, 5\le i\le n\end{cases}$$

for the algebras $\mathcal{L}_n^5,\ \mathcal{L}_n^{6,\beta}(\beta=\{1;2\})$:

$$\text{LocDer}_{\frac{1}{2}}: \begin{cases} \Delta(e_1) = \sum_{j=1}^{n} c_{j,1} e_j, \quad \Delta(e_2) = c_{2,2} e_2, \\ \Delta(e_3) = \sum_{j=2}^{n-3} c_{j,3} e_j + c_{n-1,3} e_{n-1} + c_{n,3} e_n, \\ \Delta(e_i) = \sum_{j=i}^{n} c_{j,i} e_i, \ \ 4 \le i \le n; \end{cases}$$

for the algebras $\mathcal{L}_n^{7,\gamma}(\gamma \neq 0)$, $\mathcal{L}_n^{8,\beta,\gamma}$:

$$LocDer_{\frac{1}{2}}: \begin{cases} \Delta(e_1) = \sum_{j=1}^{n} c_{j,1} e_j, \quad \Delta(e_2) = c_{2,2} e_2, \\ \Delta(e_3) = \sum_{j=2}^{n-3} c_{j,3} e_j + c_{n-1,3} e_{n-1} + c_{n,3} e_n, \\ \Delta(e_4) = c_{2,4} e_2 + \sum_{j=2}^{n-3} c_{j,4} e_j, \Delta(e_i) = \sum_{j=i}^{n} c_{j,i} e_i, \end{cases}$$

Proof. We proof the Theorem for the algebra $\mathcal{L}_n^1$, the other cases can be similarly.

Let Δ be an arbitrary local $\frac{1}{2}$-derivation on $\mathcal{L}_n^1$.

By the definition of local $\frac{1}{2}$-derivation, for any element $x \in \mathcal{L}_n^1$, there exists a $\frac{1}{2}$-derivation D_x on such $\mathcal{L}_n^1$ that $\Delta(x) = D_x(x)$.

By Proposition 3.1, the matrix of $\frac{1}{2}$-derivation D_x has the following form:

$$A_x = \begin{pmatrix} a_1^x & 0 & 0 & 0 & \dots & 0 \\ a_2^x & a_1^x & b_2^x & 0 & \dots & 0 \\ a_3^x & 0 & b_3^x & 0 & \dots & 0 \\ a_4^x & 0 & b_4^x & \frac{1}{2}(a_1^x + b_3^x) & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots & 0 \\ a_n^x & 0 & b_n^x & \frac{b_{n-1}^x}{2} & \dots & \left(1 - \frac{1}{2^{n-3}}\right) a_1^x + \frac{1}{2^{n-3}} b_3^x \end{pmatrix}$$

Let C be the matrix of Δ and

$$C = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} & \cdots & c_{2,n} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} & \cdots & c_{3,n} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} & \cdots & c_{4,n} \\ c_{5,1} & c_{5,2} & c_{5,3} & c_{5,4} & \cdots & c_{5,n} \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & c_{n-1,4} & \cdots & c_{n-1,n} \\ c_{n,1} & c_{n,2} & c_{n,3} & c_{n,4} & \cdots & c_{n,n} \end{pmatrix}$$

Then, by choosing subsequently $x = e_1$, $x = e_2, \dots, x = e_n$ and using $\Delta(x) = D_x(x)$, i.e. $C\bar{x} = A_x\bar{x}$ where $\bar{x} = (x_1, x_2, \dots, x_n)^T$ is the vector corresponding to $x = x_1e_1 + x_2e_2 + \dots + x_ne_n$, we have

$$C = \begin{pmatrix} c_{1,1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ c_{2,1} & c_{2,2} & c_{2,3} & 0 & \cdots & 0 & 0 \\ c_{3,1} & 0 & c_{3,3} & 0 & \cdots & 0 & 0 \\ c_{4,1} & 0 & c_{4,3} & c_{4,4} & \cdots & 0 & 0 \\ c_{5,1} & 0 & c_{5,3} & c_{5,4} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ c_{n-1,1} & 0 & c_{n-1,3} & c_{n-1,4} & \cdots & c_{n-1,n-1} & 0 \\ c_{n,1} & 0 & c_{n,3} & c_{n,4} & \cdots & c_{n,n-1} & c_{n,n} \end{pmatrix}.$$

Now we prove that the linear operator, defined by the matrix C is a local $\frac{1}{2}$-derivation. If, for each element $x \in \mathcal{L}_n^1$, there exists a matrix A_x of the form in Proposition 3.1 such that

$$C\bar{x} = A_x\bar{x}, \qquad (3.1)$$

then the linear operator, defined by the matrix C is a local $\frac{1}{2}$-derivation. In other words, if, for each element $x \in \mathcal{L}_n^1$, the system of equations

$$\begin{cases} c_{1,1}x_1 = a_1^x x_1, c_{2,1}x_1 + c_{2,2}x_2 + c_{2,3}x_3 = a_2^x x_1 + a_1^x x_2 + b_2^x x_3, \\ c_{3,1}x_1 + c_{3,3}x_3 = a_3^x x_1 + b_3^x x_3, \\ c_{i,1}x_1 + \sum_{j=3}^{i} c_{i,j}x_j = a_i^x x_1 + b_i^x x_3 + \\ + \left(\left(1 - 2^{3-i}\right)a_1^x + 2^{3-i}b_3^x\right)x_i + \sum_{j=4}^{i-1} 2^{3-j} b_{i-j+3}^x x_j \end{cases}$$

obtained from (3.1), has a solution with respect to the variables

$$\boldsymbol{a_1^x, a_2^x, \dots, a_n^x, b_2^x, b_3^x, \dots, b_n^x}$$

then the linear operator, defined by the matrix C, is a local $\frac{1}{2}$-derivation.

Let us consider the following cases

Case 1. Let $\boldsymbol{x_1 \neq 0}$. Then, $4 \le i \le n$

$$\boldsymbol{a_1^x = c_{1,1}, a_2^x = \frac{c_{2,1}x_1 + c_{2,2}x_2 + c_{2,3}x_3 - a_1^x x_2 - b_2^x x_3}{x_1}},$$

$$\boldsymbol{a_3^x = \frac{c_{3,1}x_1 + c_{3,3}x_3 - b_3^x x_3}{x_1}},$$

$$\boldsymbol{a_i^x = \frac{\sum_{j=3}^{i} c_{i,j}x_j - b_i^x x_3 - \left((1 - 2^{3-i})a_1^x + 2^{3-i}b_3^x\right)x_i - \sum_{j=4}^{i-1} 2^{3-j} b_{i-j+3}^x x_j}{x_1}},$$

where b_t^x, $2 \le t \le n$ defined arbitrary.

Case 2. Let $\boldsymbol{x_1 = 0}$, $\boldsymbol{x_2 \neq 0}$. Then

$$\boldsymbol{a_1^x = \frac{c_{2,2}x_2 + c_{2,3}x_3 - b_2^x x_3}{x_2}}$$

where b_2^x defined arbitrary.

Case 3. Let $\boldsymbol{x_1 = x_2 = 0, x_3 \neq 0}$. Then $\boldsymbol{4 \le i \le n}$

$$\boldsymbol{b_2^x = c_{2,2}}, \quad \boldsymbol{b_3^x = c_{3,3}},$$

$$\boldsymbol{b_i^x} = \frac{\sum_{j=3}^{i} c_{i,j}x_j - \left((1 - 2^{3-i})a_1^x + 2^{3-i}b_3^x\right)x_i}{x_3} - \frac{\sum_{j=4}^{i-1} 2^{3-j} b_{i-j+3}^x x_j}{x_3}$$

Case 4. Let $\boldsymbol{x_1 = x_2 = \ldots = x_{t-1} = 0}$, $\boldsymbol{x_t \neq 0}$, $\boldsymbol{4 \le t \le n}$. Then

$$\boldsymbol{b_3^x = 2^{t-3} c_{t,t}},$$

$$\boldsymbol{b_{i-t+3}^x = 2^{t-3} \cdot \frac{\sum_{j=t}^{i} c_{i,j}x_j - \sum_{j=t+1}^{i} b_{i-j+3}^x x_j}{x_t}, t + 1 \le i \le n},$$

where a_1^x defined arbitrary.

Corollary 3.3. The naturally graded quasi-filiform Leibniz algebras of type II admit a local $\frac{1}{2}$-derivations which are not a $\frac{1}{2}$-derivations.

CONCLUSION

In the present paper we consider $\frac{1}{2}$-derivation and local $\frac{1}{2}$-derivation on naturally graded quasi-filiform Leibniz algebras of Type II. We give the description of $\frac{1}{2}$-derivations on naturally graded quasi-filiform Leibniz algebras of Type II. Moreover, we describe the local $\frac{1}{2}$-derivation on naturally quasi-filiform Leibniz algebras of Type II. We prove that naturally graded quasi-filiform Leibniz algebras of Type II admit a local $\frac{1}{2}$-derivation which are not a $\frac{1}{2}$-derivation.

REFERENCES

1. J.Q.Adashev and B.B.Yusupov, Local derivations and automorphisms of direct sum null-filiform Leibniz algebras, Lobachevskii Journal of Mathematics, 43 (12), 2022, p. 1–7.
2. J.Q.Adashev and B.B.Yusupov, Local derivation of naturally graded quasi-filiform Leibniz algebras, Uzbek Mathematical Journal, Volume 65, Issue 1, 2021, p. 28–41.
3. A.K.Alauadinov, B.B.Yusupov, Local derivations of conformal Galilei algebra, Communications in Algebra, 52:6 (2024), 2489–2508.
4. A. K. Alauadinov, B. B. Yusupov, Local derivation on the Schrödinger Lie algebra in $n+1$-dimensional space-time. (2024), arXiv:2402.07576.
5. Sh.A.Ayupov and K.K.Kudaybergenov, Local derivations on finite-dimensional Lie algebras, Linear Algebra Appl., 493, 2016, p. 381-398.
6. Sh.A. Ayupov, A.Elduque, K.K.Kudaybergenov, Local derivations and automorphisms of Cayley algebras, Journal of Pure and Applied Algebra, 227(5), 2022, p. 107277.
7. Sh.A.Ayupov, K.K.Kudaybergenov, B.B.Yusupov, Local and 2-Local Derivations of Locally Simple Lie Algebras, Journal of Mathematical Sciences, 278: 4 (2024), p.613–622.
8. Sh.A.Ayupov, K.K.Kudaybergenov, B.B.Yusupov, Local and 2-local derivations of p-filiform Leibniz algebras, Journal of Mathematical Sciences, 245 (3), 2020, p. 359-367.
9. Sh.A.Ayupov, A.Kh.Khudoyberdiyev, B.B.Yusupov, Local and 2-local derivations of solvable Leibniz algebras, Internat. J. Algebra Comput., 30(6), 2020, p.1185-1197.
10. L.M.Camacho, J.R.Gómez, A.J.González, B.A.Omirov, Naturally graded quasi-filiform Leibniz algebra, Journal of Symbolic Computation, 44(5), 2010, p. 527-539.
11. Y.Chen, K.Zhao, Y.Zhao, Local derivations on Witt algebras, Linear and multilinear algebra, 70(6), 2022, p. 1159-1172.
12. V. Filippov, On δ-derivations of Lie algebras. Siberian Mathematical Journal, 39 (1998), 6, 1218-1230.
13. V. Filippov, δ-derivations of prime Lie algebras. Siberian Mathematical Journal, 40 (1999), 1, 174-184.
14. V. Filippov, On δ-derivations of prime alternative and Malcev algebras. Algebra and Logic, 39 (2000), 5, 354-358.
15. R.V.Kadison, Local derivations, J. Algebra, 130, 1990, p. 494-509.
16. K.K.Kudaybergenov, B.A.Omirov, T.K.Kurbanbaev, Local derivations on solvable Lie algebras of maximal rank, Communications in Algebra 50(9), 2022, p. 1-11.
17. A.Kh.Khudoyberdiyev and B.B.Yusupov, Local and 2-local $\frac{1}{2}$-derivation on finite-dimensional Lie algebras. Results in Mathematics, 79, 210 (2024).
18. D.R.Larson and A.R.Sourour, Local derivations and local automorphisms of $B(X)$, Proc. Sympos. Pure Math., 51, 1990, p. 187-194.
19. Y.Yu and Zh.Chen, Local derivations on Borel subalgebras of finite-dimensional simple Lie algebras, Comm. Algebra, 48(1), 2020, p. 1-10.
20. B. Yusupov, N. Vaisova and T. Madrakhimov, Local $\frac{1}{2}$-derivation on of n-dimensional naturally graded quasi-filiform Leibniz algebra of type I, AIP Conference Proceedings, 3147(1) (2024), 020008.