Note for the Prime Gaps

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Article

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Abstract: A prime gap is the difference between two successive prime numbers. The nth prime gap, denoted g_n is the difference between the (n + 1)st and the nth prime numbers, i.e. $g_n = p_{n+1} - p_n$. There isn't a verified solution to Andrica's conjecture yet. The conjecture itself deals with the difference between the square roots of consecutive prime numbers. While mathematicians have proven it true for a vast number of primes, a general solution remains elusive. The Andrica's conjecture is equivalent to say that $g_n < 2 \cdot \sqrt{p_n} + 1$ holds for all n. In this note, using the divergence of the infinite sum of the reciprocals of all prime numbers, we prove that the Andrica's conjecture is true.

Keywords: prime gaps; prime numbers; natural logarithm; infinite sum

MSC: 11A41; 11A25

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1. Introduction

Prime numbers, the building blocks of integers, have fascinated mathematicians for centuries. Their irregular distribution, with gaps of seemingly random size between them, is a source of ongoing intrigue. Andrica's conjecture tackles this very irregularity, proposing a relationship between the sizes of these prime gaps and the primes themselves. Andrica's conjecture (named after Dorin Andrica) is a conjecture regarding the gaps between prime numbers [1]. The conjecture states that the inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

holds for all n, where p_n is the nth prime number. If $g_n = p_{n+1} - p_n$ denotes the nth prime gap, then Andrica's conjecture can also be rewritten as

$$g_n < 2 \cdot \sqrt{p_n} + 1.$$

Imran Ghory has used data on the largest prime gaps to confirm the conjecture for n up to $1.3002 \cdot 10^{16}$ [2].

Legendre's conjecture, proposed by Adrien-Marie Legendre, states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n [2]. The conjecture is one of Landau's problems (1912) on prime numbers. If Legendre's conjecture is true, the gap between any prime p and the next largest prime would be $O(\sqrt{p}\,)$, as expressed in big O notation. Oppermann's conjecture is another unsolved problem in mathematics on the distribution of prime numbers [2]. It is closely related to but stronger than Legendre's conjecture and Andrica's conjecture. It is named after Danish mathematician Ludvig Oppermann, who announced it in an unpublished lecture in March 1877 [3]. If the conjecture is true, then the gap size would be on the order of $g_n < \sqrt{p_n}$.

This seemingly simple statement has profound implications for our understanding of prime number distribution. Unfortunately, despite its apparent elegance, Andrica's conjecture remains unproven. Mathematicians have extensively verified it for a tremendous number of primes, but a universal solution proving its truth for all primes continues to be elusive. This lack of proof doesn't diminish the significance of the conjecture. It serves as a guidepost, nudging mathematicians towards a deeper understanding of prime number distribution. The quest to solve Andrica's conjecture pushes the boundaries of our knowledge and holds the potential to unlock new insights into the enigmatic world of primes.

Whether the Andrica's conjecture is true or not still remains as an open question. By employing the divergence of the infinite sum of the reciprocals of all prime numbers and delving into the properties of natural logarithm, we will demonstrate a crucial contradiction under the assumption that there exists at least one prime gap g_{n_0} such that $g_{n_0} \geq 2 \cdot \sqrt{p_{n_0}} + 1$. This contradiction will definitively prove the Andrica's conjecture. In this way, we provide a new step forward that could help us to find a better upper bound for prime gaps.

2. Materials and methods

We know the following inequality about the natural logarithm:

Proposition 1. *For $x > -1$ [4]:*

$$\frac{x}{1+x} \leq \log(1+x) \leq x.$$

Therefore, we can prove the following Lemma:

Lemma 1. *For $x > 1$:*

$$\frac{1}{x} \leq \log\left(\frac{x}{x-1}\right) \leq \frac{1}{x-1}.$$

Proof. We know that

$$\log\left(\frac{x}{x-1}\right) = \log\left(\frac{x-1+1}{x-1}\right) = \log\left(1+\frac{1}{x-1}\right).$$

By Proposition 1, we have

$$\log\left(1+\frac{1}{x-1}\right) \leq \frac{1}{x-1}.$$

Moreover, we see that

$$\begin{aligned}
\log\left(1+\frac{1}{x-1}\right) &\geq \frac{\frac{1}{x-1}}{1+\frac{1}{x-1}} \\
&= \frac{1}{(x-1)\cdot\left(1+\frac{1}{x-1}\right)} \\
&= \frac{1}{(x-1)+1} \\
&= \frac{1}{x}
\end{aligned}$$

by Proposition 1. □

It is well-known the Mertens' second theorem:

Proposition 2. *Mertens' second theorem is*

$$\lim_{n\to\infty}\left(\sum_{p\leq n}\frac{1}{p} - \log\log n - B\right) = 0,$$

where $B \approx 0.261497$ is the Meissel-Mertens constant [5].

Using the Mertens' second theorem, we can prove the following result:

Lemma 2. *We define the following infinite sum*

$$\sum_{p \geq x} \log\left(\frac{p}{p-1}\right)$$

over all prime numbers p that are greater than or equal to x. This infinite sum diverges.

Proof. By Lemma 1, we notice that

$$\begin{aligned}\sum_{p \geq x} \log\left(\frac{p}{p-1}\right) &\geq \sum_{p \geq x} \frac{1}{p} \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{p_n}\right) - \sum_{p < x} \frac{1}{p} \\ &\geq \left(\sum_{n=1}^{\infty} \frac{1}{p_n}\right) - \frac{x}{2}\end{aligned}$$

where p_n is the nth prime. By Proposition 2, the expression

$$\left(\sum_{n=1}^{\infty} \frac{1}{p_n}\right) - \frac{x}{2}$$

diverges for every real number x. □

The following limit was found using the mathematical computation of the "Wolfram | Alpha: Computational Intelligence" web site:

Proposition 3. *We have [6]:*

$$\lim_{x \to \infty} \frac{-\sqrt[x]{a} + \sqrt[x]{b}}{-\sqrt[x]{b} + \sqrt[x]{c}} = \frac{\log(a) - \log(b)}{\log(b) - \log(c)}.$$

The following is a key Lemma.

Lemma 3. *If there exists a natural number n_0 such that*

$$\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}} > 1$$

then

$$\sqrt{p_{n_0+2}} - \sqrt{p_{n_0+1}} > 1.$$

Proof. Suppose that

$$\sqrt{p_{n_0+2}} - \sqrt{p_{n_0+1}} < 1$$

under the assumption that

$$\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}} > 1.$$

Putting both together yields

$$\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}} > \sqrt{p_{n_0+2}} - \sqrt{p_{n_0+1}}$$

that is

$$\frac{\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}}}{\sqrt{p_{n_0+2}} - \sqrt{p_{n_0+1}}} > 1.$$

We can show that

$$\frac{\sqrt{p_{n_0+1}}-\sqrt{p_{n_0}}}{\sqrt{p_{n_0+2}}-\sqrt{p_{n_0+1}}}=\frac{\sqrt[4]{p_{n_0+1}}+\sqrt[4]{p_{n_0}}}{\sqrt[4]{p_{n_0+2}}+\sqrt[4]{p_{n_0+1}}}\cdot\frac{\sqrt[4]{p_{n_0+1}}-\sqrt[4]{p_{n_0}}}{\sqrt[4]{p_{n_0+2}}-\sqrt[4]{p_{n_0+1}}}$$

such that

$$\frac{\sqrt[4]{p_{n_0+1}}+\sqrt[4]{p_{n_0}}}{\sqrt[4]{p_{n_0+2}}+\sqrt[4]{p_{n_0+1}}}<1.$$

Next, let's take the new fraction

$$\frac{\sqrt[4]{p_{n_0+1}}-\sqrt[4]{p_{n_0}}}{\sqrt[4]{p_{n_0+2}}-\sqrt[4]{p_{n_0+1}}}=\frac{\sqrt[8]{p_{n_0+1}}+\sqrt[8]{p_{n_0}}}{\sqrt[8]{p_{n_0+2}}+\sqrt[8]{p_{n_0+1}}}\cdot\frac{\sqrt[8]{p_{n_0+1}}-\sqrt[8]{p_{n_0}}}{\sqrt[8]{p_{n_0+2}}-\sqrt[8]{p_{n_0+1}}}$$

such that

$$\frac{\sqrt[8]{p_{n_0+1}}+\sqrt[8]{p_{n_0}}}{\sqrt[8]{p_{n_0+2}}+\sqrt[8]{p_{n_0+1}}}<1.$$

If we continue this iteration on and on, we arrive at:

$$1>\underbrace{\frac{\sqrt[4]{p_{n_0+1}}+\sqrt[4]{p_{n_0}}}{\sqrt[4]{p_{n_0+2}}+\sqrt[4]{p_{n_0+1}}}\cdot\frac{\sqrt[8]{p_{n_0+1}}+\sqrt[8]{p_{n_0}}}{\sqrt[8]{p_{n_0+2}}+\sqrt[8]{p_{n_0+1}}}\cdot\frac{\sqrt[16]{p_{n_0+1}}+\sqrt[16]{p_{n_0}}}{\sqrt[16]{p_{n_0+2}}+\sqrt[16]{p_{n_0+1}}}\cdot\ldots}_{\infty}.$$

Using the Proposition 3, we can calculate the remaining fraction whenever this iteration prolongs to infinity:

$$\lim_{x\to\infty}\frac{-\sqrt[x]{p_{n_0}}+\sqrt[x]{p_{n_0+1}}}{-\sqrt[x]{p_{n_0+1}}+\sqrt[x]{p_{n_0+2}}}=\frac{\log(p_{n_0})-\log(p_{n_0+1})}{\log(p_{n_0+1})-\log(p_{n_0+2})}.$$

The inequality

$$\sqrt{p_{n_0+1}}-\sqrt{p_{n_0}}>1$$

can be transformed into

$$1-\frac{1}{\sqrt{p_{n_0+1}}}>\frac{\sqrt{p_{n_0}}}{\sqrt{p_{n_0+1}}}$$

and

$$\log\left(1-\frac{1}{\sqrt{p_{n_0+1}}}\right)^2>\log(p_{n_0})-\log(p_{n_0+1})$$

by properties of natural logarithms. In addition, the inequality

$$\sqrt{p_{n_0+2}}-\sqrt{p_{n_0+1}}<1$$

can be transformed into

$$1-\frac{1}{\sqrt{p_{n_0+2}}}<\frac{\sqrt{p_{n_0+1}}}{\sqrt{p_{n_0+2}}}$$

and

$$\log\left(1-\frac{1}{\sqrt{p_{n_0+2}}}\right)^2<\log(p_{n_0+1})-\log(p_{n_0+2})$$

by properties of natural logarithms. Hence, it is enough to show that

$$\frac{\log(p_{n_0}) - \log(p_{n_0+1})}{\log(p_{n_0+1}) - \log(p_{n_0+2})} < \frac{\log\left(1 - \frac{1}{\sqrt{p_{n_0+1}}}\right)^2}{\log\left(1 - \frac{1}{\sqrt{p_{n_0+2}}}\right)^2}$$
$$= \frac{\log\left(1 - \frac{1}{\sqrt{p_{n_0+1}}}\right)}{\log\left(1 - \frac{1}{\sqrt{p_{n_0+2}}}\right)}$$
$$< 1.$$

Certainly, we deduce that

$$\log\left(1 - \frac{1}{\sqrt{p_{n_0+2}}}\right) > \log\left(1 - \frac{1}{\sqrt{p_{n_0+1}}}\right)$$

because of

$$1 - \frac{1}{\sqrt{p_{n_0+2}}} > 1 - \frac{1}{\sqrt{p_{n_0+1}}}$$

is trivially true. At the end, the product of infinite numbers that are all between 0 and 1 produces a result that is lesser than 1. Since that implies the inequality $1 > 1$ by transitivity, we reach a contradiction. Based on a proof by contradiction, we are able to affirm that

$$\sqrt{p_{n_0+2}} - \sqrt{p_{n_0+1}} > 1.$$

□

Putting all together, we show that the Andrica's conjecture is true.

3. Results

This is the main theorem.

Theorem 1. *The Andrica's conjecture is true.*

Proof. There is not any natural number n' such that

$$\sqrt{p_{n'+1}} - \sqrt{p_{n'}} = 1$$

since this implies that $g_{n'} = 2 \cdot \sqrt{p_{n'}} + 1$. For every n, g_n is a natural number and $2 \cdot \sqrt{p_n} + 1$ is always irrational. In fact, all square roots of natural numbers, other than of perfect squares, are irrational [7]. So, there exists a natural number n_0 such that

$$\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}} > 1$$

under the assumption that the Andrica's conjecture is false. By Lemma 3, we are capable to assure that

$$\sqrt{p_{n+1}} - \sqrt{p_n} > 1$$

holds for all $n \geq n_0$ whenever

$$\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}} > 1.$$

That is the same as

$$\sqrt{p_{n+1}} - 1 > \sqrt{p_n}$$

which is
$$\frac{1}{\sqrt{p_{n+1}}-1} < \frac{1}{\sqrt{p_n}}$$
for all $n \geq n_0$. This is equivalent to
$$\frac{\sqrt{p_{n+1}}}{\sqrt{p_{n+1}}-1} < \frac{\sqrt{p_n}+1}{\sqrt{p_n}}$$
after adding the number 1 to the both sides and simplifying the terms when
$$\frac{\sqrt{p_{n+1}}}{\sqrt{p_{n+1}}-1} = \left(1 + \frac{1}{\sqrt{p_{n+1}}-1}\right)$$
and
$$\frac{\sqrt{p_n}+1}{\sqrt{p_n}} = \left(1 + \frac{1}{\sqrt{p_n}}\right).$$
Now, let's try to solve the equation
$$\frac{\sqrt{p_n}}{\sqrt{p_n}-1} \cdot x = \frac{\sqrt{p_n}+1}{\sqrt{p_n}}.$$
We solve this equation as
$$\begin{aligned} x &= \frac{\sqrt{p_n}+1}{\sqrt{p_n}} \cdot \frac{\sqrt{p_n}-1}{\sqrt{p_n}} \\ &= \frac{(\sqrt{p_n}+1)\cdot(\sqrt{p_n}-1)}{\sqrt{p_n}\cdot\sqrt{p_n}} \\ &= \frac{p_n-1}{p_n}. \end{aligned}$$
In this way, we obtain that
$$\frac{\sqrt{p_{n+1}}}{\sqrt{p_{n+1}}-1} < \frac{\sqrt{p_n}}{\sqrt{p_n}-1} \cdot \frac{p_n-1}{p_n}$$
which is
$$\frac{p_n}{p_n-1} < \frac{\sqrt{p_n}}{\sqrt{p_n}-1} \cdot \frac{\sqrt{p_{n+1}}-1}{\sqrt{p_{n+1}}}$$
after distributing the terms. Finally, we obtain that
$$\log\left(\frac{p_n}{p_n-1}\right) < \log\left(\frac{\sqrt{p_n}}{\sqrt{p_n}-1}\right) - \log\left(\frac{\sqrt{p_{n+1}}}{\sqrt{p_{n+1}}-1}\right)$$
after of applying the natural logarithm. For the previous inequality, we deduce that
$$\sum_{n=n_0}^{\infty} \log\left(\frac{p_n}{p_n-1}\right) < \sum_{n=n_0}^{\infty} \left(\log\left(\frac{\sqrt{p_n}}{\sqrt{p_n}-1}\right) - \log\left(\frac{\sqrt{p_{n+1}}}{\sqrt{p_{n+1}}-1}\right)\right).$$

We check that

$$
\begin{aligned}
&\sum_{n=n_0}^{\infty}\left(\log\left(\frac{\sqrt{p_n}}{\sqrt{p_n}-1}\right)-\log\left(\frac{\sqrt{p_{n+1}}}{\sqrt{p_{n+1}}-1}\right)\right)\\
&=\log\left(\frac{\sqrt{p_{n_0}}}{\sqrt{p_{n_0}}-1}\right)+\sum_{n=n_0+1}^{\infty}\left(\log\left(\frac{\sqrt{p_n}}{\sqrt{p_n}-1}\right)-\log\left(\frac{\sqrt{p_n}}{\sqrt{p_n}-1}\right)\right)\\
&=\log\left(\frac{\sqrt{p_{n_0}}}{\sqrt{p_{n_0}}-1}\right)\\
&\leq\frac{1}{\sqrt{p_{n_0}}-1}
\end{aligned}
$$

by Lemma 1. While the fraction $\frac{1}{\sqrt{p_{n_0}}-1}$ is a real number, the following infinite sum

$$\sum_{n=n_0}^{\infty}\log\left(\frac{p_n}{p_n-1}\right)=\sum_{p\geq p_{n_0}}\log\left(\frac{p}{p-1}\right)$$

diverges by Lemma 2. Since this implies that a real number is greater than the infinity, we reach a contradiction. Consequently, by reductio ad absurdum, we conclude that the Andrica's conjecture is true. □

4. Conclusion

Further exploration about large prime gaps may involve:

- Developing new techniques in analytic number theory, the branch of mathematics that studies the distribution of prime numbers.
- Leveraging advanced computational methods to test the conjecture for even larger prime ranges and potentially uncover patterns.
- Investigating connections between Andrica's conjecture and other unsolved problems in prime number theory such as the Legendre's conjecture and Oppermann's conjecture.

To sum up, this solution for the Andrica's conjecture could be a significant advancement in our understanding of prime number distribution.

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Short Biography of Authors

Frank Vega is essentially a Back-End Programmer and Mathematical Hobbyist who graduated in Computer Science in 2007. In May 2022, The Ramanujan Journal accepted his mathematical article about the Riemann hypothesis. The article "Robin's criterion on divisibility" makes several significant contributions to the field of number theory. It provides a proof of the Robin inequality for a large class of integers, and it suggests new directions for research in the area of analytic number theory.