A Methodological Note on the Study of Queuing Networks

Manuel Alberto M. Ferreira and Marina Andrade

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Prof. Dr. **MANUEL ALBERTO M. FERREIRA**
University Institute of Lisbon, BRU/UNIDE, Lisboa, Portugal
manuel.ferreira@iscte.pt

Prof. Dr. **MARINA ANDRADE**
University Institute of Lisbon, BRU/UNIDE, Lisboa, Portugal
marina.andrade@iscte.pt

ABSTRACT

The objective of this note is to present a theorem about the approximation of any distribution by a mixture of Gama distributions, that allows to consider directly infinite states space in the study of queuing networks systems.

Keywords: Networks of queues, infinite states space, Gama distribution.

1. INTRODUCTION

The usual approach to the study of queuing networks is based on the consideration of finite states space, see for instance (1,2).

The insensitivity problem[2] may be approached for systems with infinite states space, representing these systems as limits of insensible systems sequences, see again (1,2).

Alternatively the system with infinite states space may be considered directly and use the fact that any distribution may be approached by a mixture of Gama distributions, as it is shown in the following theorem, see (3).

2. THE APPROXIMATION THEOREM

Theorem 2.1

Be $F(x)$ the Distribution Function of the positive random variable X. It is possible to choose a sequence of Distribution Functions $F_m(x)$, where each term is a mixture of Gama Distributions, such that

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[2] That is, the fact that some characteristics of the network depend on the service time distribution only through its mean. When this happens it is said that those characteristics are insensible to the service time distribution.

$$\lim_{m\to\infty} F_m(x) = F(x)$$

in the whole *x* for which *F(.)* is continuous.

Dem:

Consider

$$F_m(x) = \sum_{k=1}^{\infty} \left[F\left(\frac{k}{m}\right) - F\left(\frac{k-1}{m}\right) \right] G_m^k(x), \qquad x > 0$$

where $G_m^k(.)$ is the Distribution Function of a Gama Distribution with mean k and variance $\frac{k}{m}$, being its Characteristic Function

$$\varphi_m^k(t) = 1 + k(it) + \frac{mk(mk+1)}{m^2}\frac{(it)^2}{2!} + \frac{mk(mk+1)(mk+2)}{m^3}\frac{(it)^3}{3!} + \cdots + \frac{mk(mk+1)\dots(mk+r-1)}{m^r}\frac{(it)^r}{r!} + \cdots.$$

Suppose that there are the *X* moments till the order *r*. Giving to the $\varphi_m^k(t)$ the form

$$\varphi_m^k(t) = 1 + k(it) + \frac{mk(mk+1)}{m^2}\frac{(it)^2}{2!} + \frac{mk(mk+1)(mk+2)}{m^3}\frac{(it)^3}{3!} + \cdots + \frac{mk(mk+1)\dots(mk+r-1)}{m^r}\frac{(it)^r}{r!} + o(t^r),$$
$$\text{in the neighboroud of } t = 0,$$

the Characteristic Function of $F_m(x)$ is given by

$$
\begin{aligned}
\Phi_m(t) = {} & \sum_{k=1}^{\infty}\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right]+\left(\sum_{k=1}^{\infty}\frac{k}{m}\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right]\right)(it) \\
& +\left(\sum_{k=1}^{\infty}\left(\frac{k}{m}\right)^2\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right]\right. \\
& \left.+\frac{1}{m}\sum_{k=1}^{\infty}\frac{k}{m}\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right]\right)\frac{(it)^2}{2} \\
& +\left(\sum_{k=1}^{\infty}\left(\frac{k}{m}\right)^3\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right]+\frac{3}{m}\sum_{k=1}^{\infty}\left(\frac{k}{m}\right)^2\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right]\right. \\
& \left.+\frac{2}{m^2}\sum_{k=1}^{\infty}\frac{k}{m}\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right]\right)\frac{(it)^3}{3!}+\cdots \\
& +\left(\sum_{k=1}^{\infty}\left(\frac{k}{m}\right)^r\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right]\right. \\
& +\frac{\sum_{j=1}^{r-1}j}{m}\sum_{k=1}^{\infty}\left(\frac{k}{m}\right)^{r-1}\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right]+\cdots \\
& \left.+\frac{(r-1)!}{m^{r-1}}\sum_{k=1}^{\infty}\frac{k}{m}\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right]\right)\frac{(it)^r}{r!}+o(t^r),
\end{aligned}
$$

in the neighborood of $t = 0$.

But $\sum_{k=1}^{\infty}\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right] = \lim_{n\to\infty} F\left(\frac{n}{m}\right) - F(0) = F(\infty) = 1.$

So

$$
\sum_{k=1}^{\infty}\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right] = 1
$$

and

$$
\lim_{m\to\infty}\sum_{k=1}^{\infty}\left(\frac{k}{m}\right)^j\left[F\left(\frac{k}{m}\right)-F\left(\frac{k-1}{m}\right)\right] = \int_0^{\infty} x^j dF(x),\ \ j = 1, 2, \dots, r,
$$

by the Monotone Convergence Theorem, see (4).

So

$$\lim_{m\to\infty} \Phi_m(t) = 1 + \sum_{l=1}^{r} E[X^l] + o(t^r) = \Phi(t)$$

being $\Phi(t)$ the X Characteristic Function in the neighborhood of t = 0. In consequence

$$\lim_{m\to\infty} F_m(x) = F(x) . \blacksquare$$

Observations:

- The convergence considered in this theorem, through the characteristic function, is the convergence in distribution,

- Evidently, using this approximation for the service time distribution, it is possible to consider directly infinite states space in the study of queuing networks systems.

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