Half-Explicit Runge-Kutta Lie Group Integrators for Flexible Multibody Systems

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May 22, 2024

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Introduction

The modeling and simulation of highly flexible slender structures may be based on coarse grid space discretizations which reflect the characteristic behaviour of geometrically exact beam models. In time, we get nonlinear configuration spaces that are typical of mechanical systems with large rotations. The use of Lie groups appears to be effective and has been successfully exploited before.
Half-explicit methods avoid all kinds of Newton-Raphson iterations which are a bottleneck for the efficiency of classical implicit integrators in beam analysis. The idea of half-explicit Runge-Kutta integrators was introduced by Brasey and Hairer [2], widened for higher orders of convergence by Murua and Arnold [3]. In a recent work [4], a half-explicit Runge-Kutta method on Lie groups has been developed and tested for two benchmarks of beams described by the Cosserat model. In the present work, we discuss stabilization techniques to avoid the drift-off effect for this Lie group integrator that is based on the index-2 formulation of the equations of motion.

Methodology

The Runge-Kutta Lie group integrator follows a local coordinates approach and represents the configuration variables $q \in G$ in a neighborhood of $t = t_n$ by $q(t) = q(t_n) \circ \exp\left(\tilde{\boldsymbol{\theta}}_n(t)\right)$ with local coordinates $\boldsymbol{\theta}_n$ that are defined by the equations of motion

$$\mathbf{T}(\boldsymbol{\theta}_n)\,\dot{\boldsymbol{\theta}}_n = \mathbf{v}, \quad \mathbf{M}(q)\dot{\mathbf{v}} = -\mathbf{g}(t,q,\mathbf{v}) - \mathbf{B}^\top(q)\boldsymbol{\lambda}, \quad \mathbf{0} = \boldsymbol{\Phi}(q) \tag{1}$$

with $\boldsymbol{\theta}_n(t_n) = \mathbf{0}$. Here, G denotes a nonlinear configuration space with Lie group structure, operator $\tilde{\bullet} : \mathbb{R}^k \to \mathfrak{g}$ maps $\boldsymbol{\theta}_n \in \mathbb{R}^k$ to the corresponding element in the Lie algebra $\mathfrak{g}$, $\mathbf{T}$ is the tangent operator (that is non-singular in a neighborhood of $\boldsymbol{\theta}_n = \mathbf{0}$), $\mathbf{v}$ are the velocity coordinates, $\mathbf{M}(q)$ is the mass and inertia matrix, $\mathbf{g}(t,q,\mathbf{v})$ is the vector of external and internal forces, $\boldsymbol{\Phi}(q)$ is the constraint function at position level and $\mathbf{B}(q)$ represents its gradient [1]. Following the approach of Brasey and Hairer [2], the constraint equations $\boldsymbol{\Phi}(q) = \mathbf{0}$ in (1) are substituted by their time derivative that reads on Lie groups [1]

$$\mathbf{B}(q(t))\mathbf{v}(t) = \mathbf{0} \tag{2}$$

with $q(t) = q(t_n) \circ \exp\left(\tilde{\boldsymbol{\theta}}_n(t)\right)$. These *hidden* constraints at the level of velocity coordinates are enforced in all but one stages of an explicit Runge-Kutta method resulting in a half-explicit discretization that requires the solution of systems of *linear* equations to get stage vectors $\dot{\mathbf{V}}_{ni}$, $\boldsymbol{\Lambda}_{ni}$ approximating $\dot{\mathbf{v}}(t_n + c_i h)$, $\boldsymbol{\lambda}(t_n + c_i h)$, $(i = 2, \ldots, s+1)$, see [2, 3, 4]. With at most one extra stage, these methods achieve for $p \leq 5$ the classical order p of the underlying Runge-Kutta method [4].
In Fig. 1, this convergence behaviour is illustrated by numerical test results for the roll-up maneuver of a geometrically exact beam with $G = \left(\mathbb{S}^3 \ltimes \mathbb{R}^3\right)^{N+1}$ and N the number of edges in space discretization [4]. Here, the constraints $\boldsymbol{\Phi} = \mathbf{0}$ result from the negation of shear effects. In the numerical tests, we observe constraint residuals $\|\boldsymbol{\Phi}(q_n)\|$ in the size of 10^{-11}, i.e., the integrator does *not* suffer from the drift-off effect [3] that may result from substituting $\boldsymbol{\Phi} = \mathbf{0}$ by (2). That is in line with previous experience, see [1, Section 3.6], for G being the Special Euclidean group $SE(3)$ that is covered twice by $\mathbb{S}^3 \ltimes \mathbb{R}^3$.

Drift-off effect

We obtain a different outcome in a more general setting as illustrated in the left plot of Fig. 2 by numerical test results for the Heavy top benchmark [1] on $G = SO(3) \times \mathbb{R}^3$. Here, the drift-off effect yields constraint residuals of size $\mathscr{O}(h^p)$ for a method of order $p = 5$ and h denoting the time step size. The error constant of this $\mathscr{O}(h^p)$-term may be reduced by a factor of 30 substituting the hidden constraints (2)

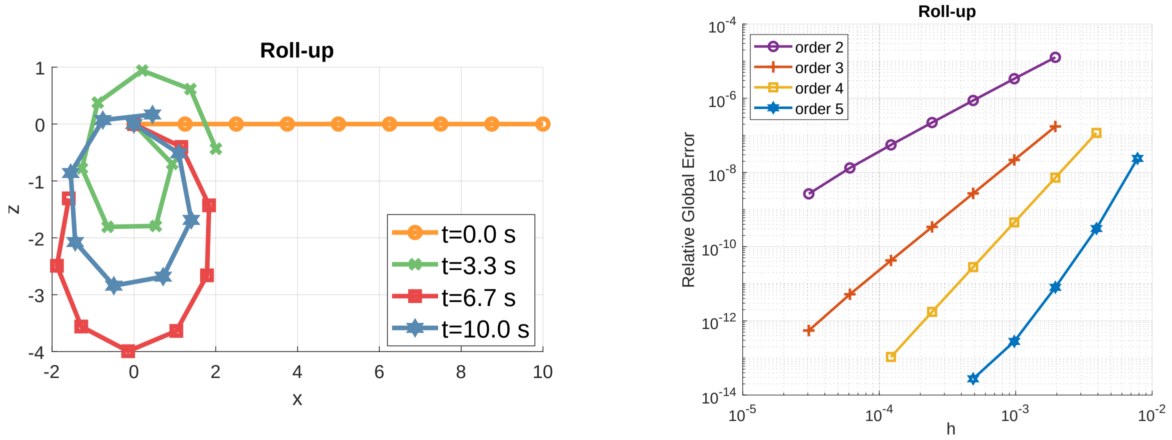

Figure 1: Half-explicit Runge-Kutta Lie group integrators applied to roll-up maneuver.

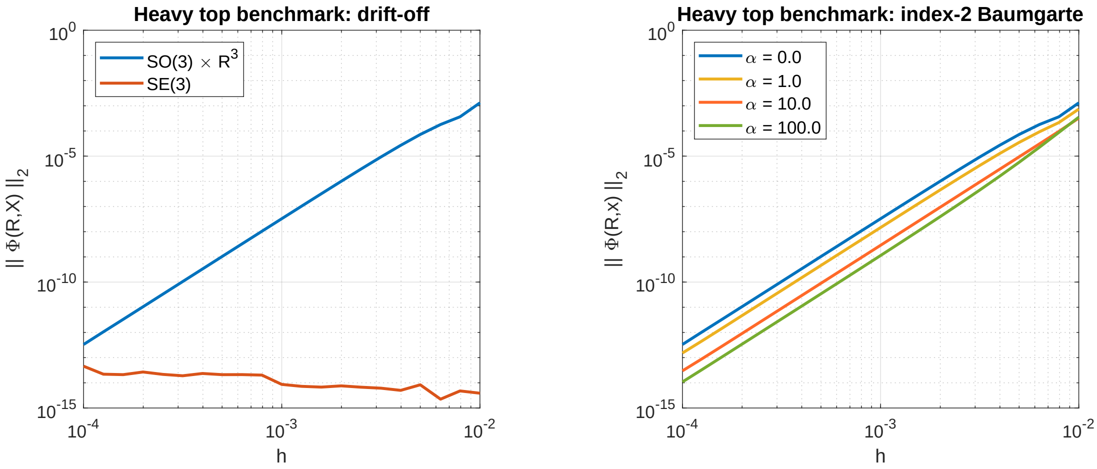

Figure 2: Half-explicit integrator of order $p = 5$ applied to heavy top benchmark. Constraint residuals.

by $\mathbf{B}(q)\mathbf{v} + \alpha\mathbf{\Phi}(q) = \mathbf{0}$ with a suitable parameter $\alpha > 0$, see Fig. 2 (right). For moderate values of α, this index-2 Baumgarte approach [5] stabilizes the evolution of constraint residuals without introducing artificial stiffness.

Alternatively, the constraints $\mathbf{\Phi}(q) = \mathbf{0}$ may be enforced by adapting classical projection techniques [3] to the Lie group setting. Let $\boldsymbol{\theta}_n^+$ denote the numerical solution of (1) at $t = t_{n+1} = t_n + h$, then $q_{n+1} \approx q(t_{n+1})$ is not just set to $q_{n+1} = q_n \circ \exp\left(\tilde{\boldsymbol{\theta}}_n^+\right)$, see [4], but defined by $q_{n+1} = q_n \circ \exp\left(\tilde{\boldsymbol{\theta}}_n^+ + \tilde{\boldsymbol{\delta}}_{\mathbf{\Phi}}\right)$ with $\boldsymbol{\delta}_{\mathbf{\Phi}} \in \mathbb{R}^k$ such that $\mathbf{\Phi}(q_{n+1}) = \mathbf{0}$ and $\mathbf{M}(q_n)\boldsymbol{\delta}_{\mathbf{\Phi}} + \mathbf{B}^\top(q_n)\boldsymbol{\mu}_n = \mathbf{0}$.

Acknowledgments

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 860124.

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