Estimation of Hurst Index and Traffic Simulation

Anatolii Pashko, Iryna Rozora and Olga Sinyavska

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ESTIMATION OF HURST INDEX AND TRAFFIC SIMULATION

Anatolii Pashko[1], Iryna Rozora[1] and Olga Syniavska[2]

[1] Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
[2] Uzgorod National University, Uzgorod, Ukraine
aap2011@ukr.net, irozora@bigmir.net, olja_sunjavska@ukr.net

Abstract. Traffic parameters of modern telecommunication networks change widely and depend on a large number of network settings, protocol characteristics and user experience. In last investigations it have been shown that the network traffic of modern networks is possessed of the property of self-similarity. And this requires finding adequate traffic simulation methods and download processes in modern telecommunications networks. Models of self-similar traffic and the process of loading telecommunication networks are based on the methods of fractional Brownian motion (FBM) simulation. The self-similarity of the fractional Brownian motion can be described by the Hurst index. In this article the methods of estimating the Hurst index and the methods of statistical modeling of fractional Brownian motion are investigated. The case of modeling with the given accuracy and reliability of fractional Brownian motion with respect of the output of a linear system is considered.

Keywords: Hurst index, self-similar traffic, fractional Brownian motion, Gaussian process.

1. Introduction

Due to the rapid development of communications and telecommunications and the emergence of new types of services, the volume of reporting information has increased sharply. Classical distributions are not always suitable for describing currently existing flows in modern networks. Therefore, new types of distributions are used to analyze traffic behavior, the study of which is not always analytically possible. One such distribution is, for example, the Pareto distribution.

Experimental and numerical studies conducted in recent decades show that traffic in many telecommunications and multimedia networks has fractal properties. Such traffic has a special structure, which is preserved when the scale is changed. For example, there are always a number of very large outliers in sample with a relatively low average level of traffic.

During the development of telecommunication networks, their intellectual analysis, statistical modeling methods are often used. The main their feature is to use simulation, performing a computational experiment of large number of times, instead of analytical methods. The simulation method is well done for problems that cannot be solved by classical mathematical methods. Applying statistical simulation, the characteristics and parameters are accumulated that reflect the behavior of complex systems, taking into account the influence of possible external factors. When using statistical modeling methods, the system is endowed with real properties and modeling is provided under different initial conditions. Accumulated data is essentially a source of data for learning.

In reality, these three tasks are needed to be solved together - Hurst index estimation, self-similar traffic modeling, traffic intelligence analysis.

The study of these properties is the purpose of this work. We continue the research presented in [1-4,34].

In the last decade, the multifractal properties of traffic have been intensively studied. These investigations were started in [5-6]. Self-similar properties of traffic allow to appear a number of traffic models based on self-similar stochastic processes [7-14]. Due to discovery of the property of self-similarity of traffic it was possible to rethink the probabilistic-temporal characteristics of such networks. Fractal or self-similar traffic models introduce concepts such as long-term dependence (the influence of the value of the number of packets that arrived some time ago on the number of packets at a given time) and self-similarity of traffic.

The complexity of the implementation of analytical and algorithmic methods for calculating networks depends significantly on the number of streams, nodes and lines. The larger these indicators, the more complex the corresponding random process and the more difficult it is to conduct a numerical analysis of the model.

The implementation of the corresponding procedures depends on the number of components in the simulated random process and on the nature of the distribution functions for the durations of the time intervals between the events. The advantage is the capabilities of computing technology. Since the speed of computers is constantly increasing, it increases the attractiveness of this method of analyzing communication networks.

Among the main characteristics of multifractal traffic is the Hurst index, which determines the degree of long-term dependence (decreasing rate in the correlation function). When evaluating the Hurst index, the properties of the obtained estimators are of great importance. Significant properties are unbiasedness, consistency.

Currently, there are many methods for estimating the Hurst parameter, but all of them are focused on such special cases of processes when the self-similarity property is combined either with a long-term dependence (fractional Brownian motion) or heavy tails.

To estimate the Hurst parameter, the most commonly used analysis is RS analysis, time analysis of variance (ANOVA) and detrended fluctuation analysis (DFA). The common property of these methods is that they are all based on the use of statistical properties of second-order samples (variance, standard deviation, correlation coefficients). In [15–16], a fractional moment method was developed, which is equally applicable for both Gaussian and heavy tails. This method can be used to estimate the Hurst parameter. All discussed methods are approximate. In [17], the estimates for the Hurst index were obtained, which are based on the use of Baxter sums and Levi-Baxter limit theorems. Generalizations of these results are given in [18].

In the article the methods are considered for estimating the Hurst index based on the use of Baxter sums and Levy-Baxter limit theorems, that improve the existing ones. The obtained estimators are objective and have all significant statistical properties.

The multifractal traffic model is usually based on random variables and processes with heavy-tailed distributions. The use of multi-fractal stochastic processes for modeling telecommunication traffic is based on fractional Brownian motion. The properties of the FBM and its practical applications were studied in [19-24].

The problem of numerical modeling of the traffic of telecommunication and computer networks is one of the main ones when creating traffic models. The result of statistical simulation is a set of realizations of a random series where the main properties of a real process are reproduced. Such models can be used in modeling telecommunication and computer networks, in the study of unfavorable network operating modes, in the numerical study of estimates of traffic characteristics in a limited sample.

Based on the simulation results, it is possible to study the influence of traffic parameters on various probabilistic-temporal characteristics. Comparative analysis allows predicting traffic behavior for various queuing algorithms, such as Primary Rate Interface (PRI), Weighted Round- Robin queue (WRR), etc.

Statistical simulation is used to check the reliability of approximate algorithms and engineering techniques for planning communication networks. When using the appropriate procedures, special attention should be paid to the tasks of determining the required number of experiments and investigating the reliability of the results obtained.

One of the important aspects of using a simulation model is to assess the accuracy and reliability of the results obtained. Methods for statistical modeling of Gaussian random processes with a given accuracy and reliability were studied in [25-30].

2. Evaluation of the Hurst index

Let (Ω, B, P) is a standard probability space.

Definition 1. We say that the Gaussian process $B_\alpha(t), t \in [0,1]$, is called the generalized Wiener process (fractional Brownian motion, FBM) with the Hurst index $\alpha \in (0,1)$ if $B_\alpha(0) = 0$, $EB_\alpha(t) = 0$ and it has a correlation function $R_\alpha(t,s) = \frac{1}{2}\left(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\right)$.

Suppose that the Hurst parameter α is unknown, such that $\alpha \in (0, \alpha^*]$, where $\alpha^* \in (0,1)$ is fixed. Also, assume that the random process $B = \{B_t,\ t \geq 0\}$ is observed at the points $\left\{\frac{k}{a_n}, 0 \le k \le a_n\right\}$, where $a_n \in \mathrm{N}$, $n \ge 1$; $a_n \to \infty$, $n \to \infty$. Suppose that for an arbitrary $\beta > 0$ the series $\sum_{n=1}^{\infty} a_n^{-\beta}$ is convergent.

Let for natural number p, $\Delta B_{k,n}^{(p)}$ be increment of order p of fractional Brownian motion B, $k = 0,..., a_n - 1$. In particular,

$$\Delta B_{k,n}^{(1)} = B_{\frac{k+1}{a_n}} - B_{\frac{k}{a_n}},$$

$$\Delta B_{k,n}^{(2)} = B_{\frac{k+2}{a_n}} - 2B_{\frac{k+1}{a_n}} + B_{\frac{k}{a_n}},$$

$$\Delta B_{k,n}^{(3)} = B_{\frac{k+3}{a_n}} - 3B_{\frac{k+2}{a_n}} + 3B_{\frac{k+1}{a_n}} - B_{\frac{k}{a_n}},$$

..................................,

$$\Delta B_{k,n}^{(p)} = \sum_{i=0}^{p} (-1)^i C_p^i B_{\frac{k}{n}+\frac{p-i}{n}}.$$

Consider the sequences of Baxter sums for fractional Brownian motion B_α:

$$\hat{S}_n^{(p)} = a_n^{2\alpha-1} \sum_{k=0}^{a_n-1} \left(\Delta B_{k,n}^{(p)}\right)^2, \; S_n^{(p)} = a_n^{2\alpha-1} \hat{S}_n^{(p)}, \; n \ge 1. \quad (1)$$

Suppose

$$V_p(k,\alpha) = \frac{1}{2} \sum_{i,j=0}^{p} (-1)^{i+j+1} C_p^i C_p^j |k + (i-j)|^{2\alpha}, \; k \ge 0, \, p \ge 1.$$

In particular,

$$V_1(0,\alpha) = 1;$$

$$V_2(0,\alpha) = 4 - 4^{\alpha};$$

$$V_3(0,\alpha) = 15 + 3^{2\alpha} - 6 \cdot 2^{2\alpha}.$$

The direct calculation makes it possible to obtain the following formulas for mathematical expectation and variance of a random variable $\hat{S}_n^p$, $p \ge 1$:

$$E\hat{S}_n^{(p)} = V_p(0,\alpha);$$

$$\mathrm{Var}\, \hat{S}_n^{(p)} = \frac{1}{n}\left(2V_p^2(0,\alpha) + 4\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right) V_p^2(k,\alpha)\right).$$

Theorem 1. Statistics

$$\hat{\alpha}_n^{(1)} = \frac{1}{2}\left(1 - \frac{\ln S_n^{(1)}}{\ln a_n}\right), \; n \ge 1$$

is a strongly consistent estimate of the Hurst parameter α.

To find the variance estimate $\mathrm{D}\,\widehat{S}_n^{(1)}$, we use the following lemma.

Theorem 2. Suppose $B = \{B_t,\ t \ge 0\}$ is fractional Brownian motion with Hurst parameter $\alpha \in (0, \alpha^*]$. Then at $\alpha^* \in (0,1)$ the following inequality holds:

$$\sup_{\alpha \in (0,\alpha^*]} \mathrm{D}\,\widehat{S}_n^{(1)} \le \frac{D_1}{a_n},$$

where

$$D_1 = \begin{cases} 2\left(3 + 2\zeta\left(4 - 4\alpha^*\right)\right), \alpha^* \in \left(0,\ \frac{3}{4}\right), \\ 2\left(3 + 2\left(1 + \ln a_n\right)\right), \alpha^* = \frac{3}{4}, \\ 2\left(3 + 2\dfrac{a_n^{4\alpha^* - 3}}{4\alpha^* - 3}\right), \alpha^* \in \left(\frac{3}{4}, 1\right), \end{cases} \qquad (2)$$

$\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s},\ s > 1$.

Similarly, it can be shown that statistics $\widehat{\alpha}_n^{(p)} = \frac{1}{2}\left(1 - \frac{\ln S_n^{(p)}}{\ln a_n}\right),\ p \ge 2$ are strongly consistent estimates of the Hurst parameter α.

Theorem 3. Suppose $B = \{B_t,\ t \ge 0\}$ is fractional Brownian motion with Hurst parameter α. Then the inequality holds:

$$\sup_{\alpha \in (0,1)} D\widehat{S}_n^{(p)} \le \frac{D_p}{a_n},$$

where

$$D_p = 2\left(K_p + \frac{1}{2}L_p + \frac{1}{2}M_p^2\zeta(4p - 4)\right),\quad p \ge 2, \qquad (3)$$

$$V_p(m,\alpha) = \frac{1}{2}\sum_{i,j=0}^{p} (-1)^{i+j+1} C_p^i C_p^j \left|m + (i - j)\right|^{2\alpha},\ k \ge 0,$$

$$K_p = \sup_{\alpha \in (0,1)} V_p^2(0,\alpha),$$

$$L_p = \sup_{\alpha \in (0,1)} V_p^2(1,\alpha),$$

$$M_p = \sup_{\alpha \in (0,1)} \left|2\alpha(2\alpha - 1)(2\alpha - 2)\cdot \ldots \cdot(2\alpha - (2p - 1))\right|,\ p \ge 2,$$

$\zeta(\cdot)$ is Riemann zeta function.

From Theorem 2 for $p = 2$ and $p = 3$, we obtain:

$V_2(0,\alpha) = 4 - 4^{\alpha};$

$$V_3(0,\alpha)=15+3^{2\alpha}-6\cdot 2^{2\alpha}.$$

and

$$V_2(1,\alpha)=7-4\cdot 2^{2\alpha}+3^{2\alpha};\ .$$

$$V_3(1,\alpha)=26-16\cdot 2^{2\alpha}+6\cdot 3^{2\alpha}-4^{2\alpha}$$

Estimates were found for p = 2

$$K_2=\sup_{\alpha\in(0,1)} V_2^2(0,\alpha)=\sup_{\alpha\in(0,1)}\left(4-4^{\alpha}\right)^2=9,$$

$$L_2=\sup_{\alpha\in(0,1)} V_2^2(1,\alpha)=\sup_{\alpha\in(0,1)}\left(7-4\cdot 2^{2\alpha}+3^{2\alpha}\right)^2=16\ ,$$

$$M_2=\sup_{\alpha\in(0,1)}\left|2\alpha(2\alpha-1)(2\alpha-2)(2\alpha-3)\right|=0{,}563\ .$$

Further, with the inequality in Theorem 2, previously obtained values K_p, L_p, M_p and considering that $\zeta(8-4\alpha)\le\zeta(4)=\frac{\pi^4}{90}$ for $\alpha\in(0,1)$, we obtain the following inequality for p = 2:

$$\sup_{\alpha\in(0,1)} D\widehat{S}_n^{(2)}\le\frac{2}{a_n}\left(9+\frac{16}{2}+\frac{1}{2}\cdot(0.563)^2\cdot\frac{\pi^4}{90}\right)\approx\frac{34.344}{a_n}\ .$$

Similarly, from the inequality in Theorem 2, previously obtained values for K_p, L_p, M_p and considering that $\zeta(12-4\alpha)\le\zeta(8)=\frac{\pi^8}{9450}$ for $\alpha\in(0,1)$, we obtain the following inequality for p = 3:

$$\sup_{\alpha\in(0,1)} D\widehat{S}_n^{(2)}\le\frac{2}{a_n}\left(100+\frac{225}{2}+\frac{1}{2}\cdot(129.738)^2\cdot\frac{\pi^8}{9450}\right)\approx\frac{1.733\cdot 10^4}{a_n}\ .$$

Theorem 4. Suppose $B=\{B_t,\ t\ge 0\}$ is fractional Brownian motion with Hurst parameter α . Then $\widehat{S}_n^{(p)}\to V_p(0,\alpha),\ p\ge 1$ with probability one as $n\to\infty$.

3. Confidence intervals

Theorem 5. The interval $\left(\widehat{\alpha}_n^{(p)}-l_\varepsilon(n),\ \widehat{\alpha}_n^{(p)}+r_\varepsilon(n)\right)\cap(0,1)$, where

$$\widehat{\alpha}_n^{(p)}=\frac{1}{2}\left(1-\frac{\ln S_n^{(p)}}{\ln a_n}\right),$$

$$l_\varepsilon(n)\ge-\frac{1}{2}\frac{\ln\left(1-\sqrt{\frac{2D_p}{a_n\varepsilon}}\right)}{\ln a_n}$$ provided that $\sqrt{\frac{2D_p}{\varepsilon}}<1$,

$$r_\varepsilon(n) \ge \frac{1}{2}\frac{\ln\left(\sqrt{\frac{2D_p}{a_n\varepsilon}}+1\right)}{\ln a_n},$$

$S_n^{(p)}$ is defined by the equality (1), D_p is determined for $p=1$ by the equality (2), and for $p \ge 2$ is defined by the equality (3), is the confidence interval for the Hurst parameter α with confidence level $1-\varepsilon \in (0,1)$.

Another estimate of the Hurst parameter α is obtained from the likelihood convergence of the unit of Baxter sums in the theorem 2 with $p=1,2,3$. Indeed, for arbitrary $\alpha \in (0,\ 1)$ there are convergence

$$\frac{\widehat{S}_n^{(2)}}{\widehat{S}_n^{(1)}} \to \frac{V_2(0,\alpha)}{V_1(0,\alpha)} = 4-4^\alpha,$$

$$\frac{\widehat{S}_n^{(3)}}{\widehat{S}_n^{(1)}} \to \frac{V_3(0,\alpha)}{V_1(0,\alpha)} = 15+3^{2\alpha}-6\cdot 2^{2\alpha},$$

$$\frac{\widehat{S}_n^{(3)}}{\widehat{S}_n^{(2)}} \to \frac{V_3(0,\alpha)}{V_2(0,\alpha)} = \frac{15+3^{2\alpha}-6\cdot 2^{2\alpha}}{4-4^\alpha}.$$

with probability 1 as $n \to \infty$.

Let's consider function $\theta_{2,1}(\alpha) = \frac{V_2(0,\alpha)}{V_1(0,\alpha)} = 4-4^\alpha$, $\alpha \in (0,1)$,

which is continuous and decreasing at interval $(0,\ 1)$, $\theta_{2,1}(0+)=3$, $\theta_{2,1}(1-)=0$. Suppose $\alpha_{2,1}(\theta)$, $\theta \in (0\ ,\ 3)$ is function, inverse to the function $\theta_{2,1}(\alpha)$, $\alpha \in (0,\ 1)$. Suppose

$$\widehat{\theta}_n = \frac{S_n^{(2)}}{S_n^{(1)}} = \frac{\widehat{S}_n^{(2)}}{\widehat{S}_n^{(1)}},\ n \ge 1. \qquad (4)$$

Then $\widehat{\theta}_n \to \theta = \theta(\alpha)$ with probability 1 as $n \to \infty$. From the following considerations it follows

Theorem 6. Statistics

$$\widehat{\alpha}_n^{(2,1)} = 1-\frac{1}{2}\log_2\left(\widehat{\theta}_n+1\right),\ n \ge 1$$

is a strongly consistent estimate of the Hurst parameter α.

Theorem 7. Suppose $\{X_k \mid 0 \le k \le a_n, a_n \in \mathrm{N}\}$, $\{Y_k \mid 0 \le k \le a_n, a_n \in \mathrm{N}\}$ are sets of random variables with finite moments of 4th order such that $EX_k = EY_k = 0,\ EX_k^2 = EX_0^2,\ EY_k^2 = EY_0^2,\ 0 \le k \le a_n;$

$$S_1 = \sum_{k=0}^{a_n} X_k^2, \qquad S_2 = \sum_{k=0}^{a_n} Y_k^2, \qquad \delta = \frac{EX_0^2}{EY_0^2}.$$

Then for arbitrary $\varepsilon > 0$ there is inequality:

$$P\left\{ \left| \frac{S_1}{S_2} - \delta \right| > \varepsilon \right\} \le \frac{\mathrm{Var}\, Q_1}{(EQ_1)^2} + \frac{\mathrm{Var}\, Q_2}{(EQ_2)^2},$$

where $Q_1 = (\delta - \varepsilon)S_2 - S_1, Q_2 = S_1 - (\delta + \varepsilon)S_2$.

Statistics $\widehat{\theta}_n$, $n \ge 1$, defined in equality (4), because Theorem 5, is a strongly consistent estimate of the parameter $\theta = \theta(\alpha)$. Using Theorem 7, we construct a confidence interval for the parameter θ, and then get the confidence interval for the Hurst parameter α. Suppose $(1-\varepsilon) \in (0,1)$ is confidence level. We define positive number $m_\varepsilon(n)$ so that

$$P\left\{ \left| \widehat{\theta}_n - \theta \right| \ge m_\varepsilon(n) \right\} \le \varepsilon.$$

For statistics $\widehat{\theta}_n = \dfrac{\widehat{S}_n^{(2)}}{\widehat{S}_n^{(1)}}$ we have:

$$Q_1 = (\theta - m_\varepsilon(n))\widehat{S}_n^{(1)} - \widehat{S}_n^{(2)} \quad \text{and} \quad Q_2 = \widehat{S}_n^{(2)} - (\theta + m_\varepsilon(n))\widehat{S}_n^{(1)}.$$

For mathematical expectations of random variables Q_1, Q_2 have:

$$EQ_1 = EQ_2 = -m_\varepsilon(n) V_1(0, H) = -m_\varepsilon(n).$$

To find upper estimate of variances of these random variables we apply inequality $(a+b)^2 \le 2(a^2 + b^2)$, $a, b \in R$ and lemma 1:

$$\mathrm{D}Q_1 \le 2(\theta - m_\varepsilon(n))^2 \,\mathrm{D}\, S_n^{(1)} + 2\mathrm{D}\, S_n^{(2)} \le \frac{2}{a_n}(\theta - m_\varepsilon(n))^2 D_1 + \frac{2}{a_n} D_2.$$

Similarly,

$$DQ_2 \le \frac{2}{a_n}(\theta + m_\varepsilon(n))^2 D_1 + \frac{2}{a_n} D_2.$$

We will select $\gamma_n(p)$ so that the following inequalities hold:

$$\frac{D\,Q_1}{(EQ_1)^2} \le \frac{2(\theta - m_\varepsilon(n))^2 D_1 + 2D_2}{m_\varepsilon^2(n) a_n} \le \frac{\varepsilon}{2},$$

$$\frac{\mathrm{D}\,Q_2}{(EQ_2)^2} \le \frac{2(\theta + m_\varepsilon(n))^2 D_1 + 2D_2}{m_\varepsilon^2(n) a_n} \le \frac{\varepsilon}{2}. \qquad (5)$$

For $\alpha^* \in (0,1)$ we have:

$$D_1 = \begin{cases} 2\left(3 + 2\zeta\left(4 - 4\alpha^*\right)\right), \alpha^* \in \left(0,\ \frac{3}{4}\right), \\ 2\left(3 + 2\left(1 + \ln a_n\right)\right), \alpha^* = \frac{3}{4}, \\ 2\left(3 + 2\frac{a_n^{4\alpha^* - 3}}{4\alpha^* - 3}\right), \alpha^* \in \left(\frac{3}{4},\ 1\right), \end{cases}$$

where $\zeta(\cdot)$ is Riemann zeta function and $D_2 \approx \dfrac{34.344}{a_n}$.

For $\alpha \in \left(0, \alpha^*\right]$, $\theta \in (0, 3)$ we have:

$$\frac{2\left(3 + m_\varepsilon(n)\right)^2 D_1 + 2D_2}{m_\varepsilon^2(n) a_n} \le \frac{\varepsilon}{2}.$$

Solve this inequality with respect to $m_\varepsilon(n)$ provided $\frac{\varepsilon}{4} a_n - D_1 > 0$, that is true for sufficiently large $n \in \mathrm{N}$:

$$2\left(3 + m_\varepsilon(n)\right)^2 D_1 + 2D_2 \le \frac{\varepsilon}{2} a_n m_\varepsilon^2(n),$$

$$\left(\frac{\varepsilon}{4} a_n - D_1\right) m_\varepsilon^2(n) - 6D_1 m_\varepsilon(n) - \left(9D_1 + D_2\right) \ge 0,$$

$$m_\varepsilon(n) \ge \frac{6D_1 + \sqrt{D}}{2\left(\frac{\varepsilon}{4} a_n - D_1\right)},$$

provided that the inequality $D = \varepsilon a_n \left(9D_1 + D_2\right) - 4D_1 D_2 \ge 0$ is true for sufficiently large n. Therefore, the following theorem holds.

Theorem 8. Suppose $\alpha \in \left(0, \alpha^*\right]$, where $\alpha^* \in (0,1)$ is fixed, $\frac{\varepsilon}{4} a_n - D_1 > 0$, $(1 - \varepsilon)$, $\varepsilon \in (0,1)$ is confidence level, D_1 is calculated by the formula (2), D_2 is calculated by the formula (3). Then the following inequality holds:

$$P\left\{H \in \left(H_{n,l}, H_{n,r}\right)\right\} \ge 1 - \varepsilon,$$

where

$$H_{n,l} = \varphi\left(\min\left(\widehat{\theta}_n + m_\varepsilon(n),\ 3\right)\right),$$

$$H_{n,r} = \varphi\left(\max\left(\theta\left(H^*\right), \widehat{\theta}_n - m_\varepsilon(n)\right)\right),$$

$$\widehat{\theta}_n = \frac{\widehat{S}_n^{(2)}}{\widehat{S}_n^{(1)}},\ m_\varepsilon(n) \ge \frac{6D_1 + \sqrt{D}}{2\left(\frac{\varepsilon}{4} a_n - D_1\right)},$$

$D = \varepsilon a_n\left(9D_1 + D_2\right) - 4D_1 D_2 \geq 0$,

$\varphi(\theta) = 1 - \frac{1}{2}\log_2(\theta+1)$, $\theta \in (0,3)$.

It's naturally and important case to study a process measured with some error. In this situation, the following results were obtained.

Assume that the values $X_\alpha(0), X_\alpha\left(\frac{1}{n}\right), ..., X_\alpha(1)$ are observed that are differed from real values of fractional Brownian motion (FBM) $\{B_\alpha(t), t \in R\}$ at the points $\left\{\frac{k}{n}\middle| 0 \leq k \leq n, n \geq 1\right\}$, on the measurement errors $\{\delta_{k,n} | 0 \leq k \leq n\}$, which are independent on the values of the FBM $\left\{B_\alpha\left(\frac{k}{n}\right)\middle| 0 \leq k \leq n\right\}$, and

$$X_\alpha\left(\frac{k}{n}\right) = B_\alpha\left(\frac{k}{n}\right) + \delta_{k,n} \qquad (6)$$

Suggest that $\delta_{k,n}$ are mutually independent identically distributed, $\delta_{k,n} \cong N\left(0, \sigma_n^2\right)$, where $\sigma_n^2 \to 0, n \to \infty$. We also assume that Hurst index α is such that $\alpha \leq \alpha^* < 1$, where the quantity α^* is known.

Let's introduce the following notations:

$\Delta^{(1)} B_{k,n} = B_\alpha\left(\frac{k+1}{n}\right) - B_\alpha\left(\frac{k}{n}\right)$,

$\Delta^{(1)} \delta_{k,n} = \delta_{k+1,n} - \delta_{k,n}$,

$\Delta^{(1)} X_{k,n} = X_\alpha\left(\frac{k+1}{n}\right) - X_\alpha\left(\frac{k}{n}\right)$, $0 \leq k \leq (n-1)$.

Consider the sequence of Baxter sum:

$S_n^{(1)}(X) = \sum_{k=0}^{n-1}\left(\Delta^{(1)} X_{k,n}\right)^2 - 2n\sigma_n^2$,

$\hat{S}_n^{(1)}(X) = n^{2\alpha-1} S_n^{(1)}(X)$, $n \geq 1$..

From the lemma for FBM $\{B_\alpha(t), t \in R\}$ as $n \to \infty$ and for all $\alpha \in (0,1)$, $k = 1, a_n = n$, the mean-square convergence holds true

$n^{2\alpha-1}\sum_{k=0}^{n-1}\left(\Delta^{(1)} B_{k,n}\right)^2 \to 1$.

Theorem 9. Let $\{B_\alpha(t), t \in R\}$ be FMB, $\alpha \in (0, \alpha^*] \subset (0,1)$ and relation (6) is fulfilled. Then

$$\sup_{\alpha \in (0,\alpha^*]} Var\left(\hat{S}_n^{(1)}(X)\right) \leq D\left(\alpha^*, n\right), \qquad (7)$$

where

$$D\left(\alpha^*, n\right) = \frac{10}{n} + 8n^{2\alpha^*-1}\sigma_n^2 + 8n^{4\alpha^*-1}\left(1-\frac{1}{n}\right)\sigma_n^4 +$$

$$+\begin{cases} \frac{2}{n}\zeta\left(4-4\alpha^*\right), & \alpha^* \in \left(0, \frac{3}{4}\right); \\ \frac{2}{n}\left(1+\ln(n)\right), & \alpha^* = \frac{3}{4}; \\ \frac{2}{n}\left(1+\frac{n^{4\alpha^*-3}}{4\alpha^*-3}\right), & \alpha^* \in \left(\frac{3}{4}, 1\right). \end{cases} \qquad (8)$$

$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s > 1$ is Riemann zeta function.

Assume that the inequality

$$\left|\hat{S}_n^{(1)}(X) - 1\right| = \left|n^{2\alpha-1}S_n^{(1)}(X) - 1\right| < \varepsilon$$

holds with large probability.

It means that inequality

$$P\left\{\left|\hat{S}_n^{(1)}(X) - 1\right| > \varepsilon\right\} \le p$$

Holds true with small probability $p \in (0,1)$.

Under made assumptions from above inequality follows the double inequality:

$$1-\varepsilon < n^{2\alpha-1}S_n^{(1)}(X) < 1+\varepsilon.$$

Hence, solving inequality above with respect to unknown parameter $\alpha \in \left(0, \alpha^*\right]$, we obtain the estimate for α.

$$\frac{1}{2}\left(1+\frac{\ln(1-\varepsilon) - \ln\left(S_n^{(1)}(X)\right)}{\ln(n)}\right) < \alpha <$$

$$< \frac{1}{2}\left(1+\frac{\ln(1+\varepsilon) - \ln\left(S_n^{(1)}(X)\right)}{\ln(n)}\right).$$

To find now the estimate for ε we use Chebyshev's inequality. Really, we obtain:

$$P\left\{\left|\hat{S}_n^{(1)}(X) - 1\right| > \varepsilon\right\} \le \frac{Var\left(\hat{S}_n^{(1)}(X) - 1\right)}{\varepsilon^2} \le p. \qquad (9)$$

We can apple obtained in lemma * the upper estimate for the value $Var\left(\hat{S}_n(X)\right)$. Then, we have

$$P\left\{\left|\hat{S}_n^{(1)}(X) - 1\right| > \varepsilon\right\} \le \frac{E\left(\hat{S}_n^{(1)}(X) - 1\right)^2}{\varepsilon^2} \le \frac{D\left(\alpha^*, n\right)}{\varepsilon^2} \le p.$$

So, we obtain the following inequality: $\varepsilon \geq \sqrt{\frac{D(\alpha^*, n)}{p}}$

Hence, under corresponding values of σ, α and under optimal number of observations *n* using inequality above the estimate of the quantity ε is found. So, the following theorem holds true.

Theorem 10. Let $\alpha \in (0, \alpha^*] \subset (0,1)$. Then $(I_l(n), I_r(n)) \cap (0,1)$ is confidence interval for Hurst index α with the level of confidence $(1-p) \in (0,1)$, where

$$I_l(n) = \frac{1}{2}\left(1 + \frac{\ln(1-\varepsilon) - \ln\left(S_n^{(1)}(X)\right)}{\ln(n)}\right),$$

$$I_r(n) = \frac{1}{2}\left(1 + \frac{\ln(1+\varepsilon) - \ln\left(S_n^{(1)}(X)\right)}{\ln(n)}\right).$$

4. Spectral representation of FBM

Process with stationary increments. The fractional Brownian motion $B_\alpha(t)$ is a process with stationary increments [19].

Then the random process $w(t) = B_\alpha(t+\Delta) - B_\alpha(t)$ with fixed Δ is a stationary Gaussian process with a correlation function

$$Ew(t+\tau)w(t) = \frac{1}{2}\left(|\tau+\Delta|^{2\alpha} + |\tau-\Delta|^{2\alpha} - 2|\tau|^{2\alpha}\right)$$

and spectral density $g(\lambda) = \frac{A^2}{\pi}\left(\frac{1-\cos(\lambda\Delta)}{|\lambda|^{2\alpha+1}}\right)$, $\lambda \in (-\infty, +\infty)$,

where $A^2 = \left(\frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda)}{\lambda^{2\alpha+1}} d\lambda\right)^{-1} = \left(-\frac{2}{\pi}\Gamma(-2\alpha)\cos(\alpha\pi)\right)^{-1}$.

Since $B_\alpha(0) = 0$ then for any Δ for the model construction of the fractional Brownian motion the following iteration relationship can be used.

$$B_\alpha(t+\Delta) = B_\alpha(t) + w(t) . \qquad (10)$$

The simulation of the fractional Brownian motion reduces to the modeling of the Gaussian stationary process. Some simulation methods for stationary Gaussian processes were studied in paper [26,27].

Let $\xi(t)$ be real-valued Gaussian stationary stochastic process with correlation function $R(\tau)$ and spectral function $F(\lambda)$, $R(\tau) = \int_0^\infty \cos(\lambda t) dF(\lambda)$.

Gaussian stationary process can be given as

$$\xi(t)=\int_0^\infty \cos(\lambda t)d\xi_1(\lambda)+\int_0^\infty \sin(\lambda t)d\xi_2(\lambda),$$

where $\xi_1(t)$ and $\xi_2(t)$ are centered uncorrelated random process such that for $0\le\lambda_1<\lambda_2$ we have

$$E(\xi_1(\lambda_2)-\xi_1(\lambda_1))^2=F(\lambda_2)-F(\lambda_1),$$
$$E(\xi_2(\lambda_2)-\xi_2(\lambda_1))^2=F(\lambda_2)-F(\lambda_1).$$

Under D_Λ we denote the partition of the interval $[0,\Lambda]$, $D_\Lambda: 0=\lambda_0<\lambda_1<\ldots<\lambda_n=\Lambda$. The model of stochastic process $\xi(t)$ can be written as $S_n(t,\Lambda)=\sum_{i=0}^{n-1}[\cos(\lambda_i t)\eta_{1i}+\sin(\lambda_i t)\eta_{2i}]$, where $\{\eta_{1i},\eta_{2i}\}$ are centered uncorrelated strictly sub-Gaussian random variables with variance $E(\eta_{1i})^2=E(\eta_{2i})^2=F(\lambda_{i+1})-F(\lambda_i)$.

It means that stochastic process $w(t)$ can be presented as follows

$$w(t)=\int_0^\infty \cos(\lambda t)d\xi_1(\lambda)+\int_0^\infty \sin(\lambda t)d\xi_2(\lambda). \quad (11)$$

For the partition $D_\Lambda: 0=\lambda_0<\lambda_1<\ldots<\lambda_n=\Lambda$, the model of stochastic process $w(t)$ is

$$w_n(t,\Lambda)=\sum_{k=0}^{n-1}(\sin(\lambda_k t)X_k+\cos(\lambda_k t)Y_k), \quad (12)$$

where $\{X_k,Y_k\}$ are uncorrelated sub-Gaussian random variables with $EX_k=EY_k=0$ and $E(X_k)^2=E(Y_k)^2=\int_{\lambda_k}^{\lambda_{k+1}} g(\lambda)d\lambda$.

The accuracy and reliability of the model in $L_2(T)$. Suppose that stochastic process $X(t)$ and all models $X_n(t,\Lambda)$ belong to some Banach space $A(T)$ зwith respect to the norm $\|\cdot\|$. We also suggest that two values $\delta>0$ i $0<\varepsilon<1$ are given.

Definition 2. The model $X_n(t,\Lambda)$ approximates the process $X(t)$ with reliability $1-\varepsilon$ and accuracy δ in the norm of the space $A(T)$, if inequality $P\{\|X(t)-X_n(t,\Lambda)\|>\delta\}\le\varepsilon.$ holds true.

Theorem 11. The model $S_n(t,\Lambda)$ will approximate the process $\xi(t)$ with reliability $1-\varepsilon$ and accuracy δ with respect to the norm of the space $L_2(T)$, if for Λ and n the inequalities $B2_{n,\Lambda} < \delta^2$ and $\exp\left\{\frac{1}{2}\right\}\frac{\delta}{\sqrt{B2_{n,\Lambda}}}\exp\left\{-\frac{\delta^2}{2B2_{n,\Lambda}}\right\} \le \varepsilon$,

are satisfied, where $B2_{n,\Lambda} = \int_T E(\xi(t) - S_n(t,\Lambda))^2 d\mu(t)$.

Теорема 12. The model $w_n(t,\Lambda)$ will approximate the process $w(t)$ with reliability $1-\varepsilon$ and accuracy δ in Banach space $L_2(T)$, if for the values Λ and n we obtain

$$B3_{n,\Lambda} < \delta^2 \text{ and } \exp\left\{\frac{1}{2}\right\}\frac{\delta}{\sqrt{B3_{n,\Lambda}}}\exp\left\{-\frac{\delta^2}{2B3_{n,\Lambda}}\right\} \le \varepsilon,$$

where $B3_{n,\Lambda} = 2T\sum_{i=0}^{n}\int_{\lambda_i}^{\lambda_{i+1}}\left(1-\frac{\sin(T(\lambda-\lambda_i))}{T(\lambda-\lambda_i)}\right)g(\lambda)d\lambda + +T\left(\int_{\Lambda}^{\infty} g(\lambda)d\lambda\right).$

Let for D_Λ we have $T(\lambda_{i+1}-\lambda_i) \le 1$. In this case the following corollary can be proved.

Theorem 13. The model $w_n(t,\Lambda)$ will approximate the process $w(t)$ with reliability $1-\varepsilon$ and accuracy δ in Banach space $L_2(T)$, if for the values Λ and n we obtain

$$G1_{n,\Lambda} < \delta^2 \text{ and } \exp\left\{\frac{1}{2}\right\}\frac{\delta}{\sqrt{G1_{n,\Lambda}}}\exp\left\{-\frac{\delta^2}{2G1_{n,\Lambda}}\right\} \le \varepsilon,$$

where $G1_{n,\Lambda} = \frac{T}{3}\sum_{i=0}^{n}\int_{\lambda_i}^{\lambda_{i+1}}(\lambda-\lambda_i)^2 g(\lambda)d\lambda + +T\left(\int_{\Lambda}^{\infty} g(\lambda)d\lambda\right).$

In the case of the partition D_Λ in a such way $\lambda_{i+1}-\lambda_i = \frac{\Lambda}{n}$ and $\frac{T\Lambda}{n} \le 1$ the result is clarified as follows. If we put $\Lambda = \left(\frac{3n^2}{T^3}\right)^{\frac{1}{2\alpha+2}}$, then

$$G2_n = T^{\frac{3\alpha}{\alpha+1}}\left(1+\frac{T}{\alpha}\right)\left[\left(3n^2\right)^{\frac{\alpha}{\alpha+1}}\right]^{-1}.$$

Теорема 14. The model $w_n(t,\Lambda)$ approximates stochastic process $w(t)$ with given reliability $1-\varepsilon$ and accuracy δ in Banach space $L_2(T)$, if for Λ and n the inequalities are satisfied

$$n > \frac{1}{\sqrt{3}}\left[T^{\frac{3\alpha}{2\varepsilon+1}}\left(1+\frac{T}{\alpha}\right)\delta^{-2}\right]^{\frac{\alpha+1}{2\alpha}}$$

and $$\exp\left\{\frac{1}{2}\right\}\frac{\delta}{\sqrt{G2_n}}\exp\left\{-\frac{\delta^2}{2G2_n}\right\} \le \varepsilon .$$

Random series. In [28] it's shown that Fractional Brownian Motion can be expanded in the form of random series]

$$B_\alpha(t) = \sum_{k=1}^{\infty}\left(a_k \sin(x_k t)X_k + b_k\left(1-\cos(y_k t)\right)Y_k\right), \qquad (13)$$

where $\{X_k, Y_k\}$ are uncorrelated standard Gaussian random variables,
$\{x_k\}$ are zeros of Bessel function $J_{-\alpha}(x)$ and
$\{y_k\}$ zeros of Bessel function $J_{1-\alpha}(x)$,

$$a_k = \frac{\pi^\alpha\sqrt{2C}}{x_k^{\alpha+1}J_{1-\alpha}(x_k)}, \qquad b_k = \frac{\pi^\alpha\sqrt{2C}}{y_k^{\alpha+1}J_{-\alpha}(y_k)}, \qquad C = \frac{\Gamma(2\alpha+1)\sin(\pi\alpha)}{\pi^{2\alpha+1}} .$$

To calculate zeros of Bessel functions we will use the following relationships

$$x_n = \left(n+\frac{3}{4}-\frac{\alpha}{2}\right)\pi - \frac{4\alpha^2-1}{2\pi(4n+3-2\alpha)} + \ldots$$

$$y_n = \left(n+\frac{5}{4}-\frac{\alpha}{2}\right)\pi - \frac{4(1-\alpha)^2-1}{2\pi(4n+1+2\alpha)} + \ldots .$$

And to compute Bessel functions it's useful to apply the next representation

$$J_{1-\alpha}^2(x_n) = \sqrt{\frac{2}{\pi x_n}}\left(\cos\left(x_n+\frac{2\alpha\pi-\pi}{4}\right) - \frac{4\alpha^2-1}{8x_n}\sin\left(x_n+\frac{2\alpha\pi-\pi}{4}\right)\right),$$

$$J_{-\alpha}^2(y_n) = \sqrt{\frac{2}{\pi y_n}}\left(\cos\left(y_n+\frac{2(1-\alpha)\pi-\pi}{4}\right) - \frac{4(1-\alpha)^2-1}{8y_n}\sin\left(y_n+\frac{2(1-\alpha)\pi-\pi}{4}\right)\right).$$

The model of stochastic process we can construct as $S_\alpha(t,M) = \sum_{k=1}^{M}\left(a_k \sin(x_k t)X_k + b_k\left(1-\cos(y_k t)\right)Y_k\right)$,

where $\{X_k, Y_k\}$ are jointly uncorrelated standard Gaussian random variables.

Simulating independent Gaussian random variables we can obtain the wider class of random variables, namely, strictly sub-Gaussian random variables due to the accuracy of representation and calculation of real numbers. The zeros of the Bessel functions and the values are also calculated with some accuracy.

We will denote by $\breve{a}_k, \breve{b}_k, \breve{x}_k, \breve{y}_k$ the approximate values of a_k, b_k, x_k, y_k.

Let $\left|a_k - \breve{a}_k\right| \le h_k^a$, $\quad \left|b_k - \breve{b}_k\right| \le h_k^b$,

$\left|x_k - \breve{x}_k\right| \le h_k^x$, $\quad \left|y_k - \breve{y}_k\right| \le h_k^y$,

Where $h_k^a, h_k^b, h_k^x, h_k^y$ are predetermined accuracy.

Then the model of Fractional Brownian Motion will be as follows

$$\breve{S}_\alpha(t,M)=\sum_{k=1}^{M}\left(\breve{a}_k\sin(\breve{x}_k t)X_k+\breve{b}_k\left(1-\cos(\breve{y}_k t)\right)Y_k\right).$$

The accuracy of the modeling $\Delta(t)$ equals $\Delta(t)=B_\alpha(t)-\breve{S}_\alpha(t,M)$.

Applying this model it's necessary to compute the zeros of Bessel functions with high quality/ It needs a great preprocessing and preparing efforts.

Теорема 15. The model $\breve{S}_\alpha(t,M)$ will approximate the process $W_\alpha(t)$ with accuracy $\delta>0$ and reliability $1-\varepsilon,\ \ 0<\varepsilon<1$ in the norm of the space $L_2([0,T])$, if the following relationships

$$\delta^2>B1_M, \qquad \text{and} \qquad \exp\left\{\frac{1}{2}\right\}\frac{\delta}{\sqrt{B1_M}}\exp\left\{-\frac{\delta^2}{2B1_M}\right\}\le\varepsilon,$$

are fulfilled

where $B1_M=T\sum_{k=M+1}^{\infty}\left(a_k^2+4b_k^2\right)+.T\sum_{k=1}^{M}\left(\left(Ta_k h_k^x+h_k^a\right)^2+\left(Tb_k h_k^y+2h_k^b\right)^2\right)$.

5. The estimation of traffic volume and load process

Under $A(t)$ we denote the amount of traffic coming to the network over a period of time $[0,T]$. The increment we denote as $A(s,t)=A(t)-A(s),\quad t>s>0$. In [5] it's shown that input traffic has a view $A(t)=mt+\sqrt{am}W_\alpha(t)$, where m is an average traffic rate, $W_\alpha(t)$ - FBM with Hurst index $\alpha\in\left(\frac{1}{2},1\right)$, a is some constant.

If the network has one device with the rate of service $C>m$, the process of loading is determined [6] by the formula $Q(t)\cong\sup_{s\le t}\left(A(s,t)-C(t-s)\right)$. Having n independent identically loading device we obtain $Q_n(t)\cong\sup_{s\le t}\left(\sum_{i=1}^{n}A_i(s,t)-nC(t-s)\right)$. Here, the symbol $\cong$ means identically distributed quantity.

Investigate now the probability of overloading by threshold b of $Q(t)$ on time interval $[0,T]$. We will denote $Q\cong\sup_{t\in[0,T]}\left(Q(t)\right),\quad \pi(b)=P\{Q\ge b\}$.

We find the upper estimate of the overloading probability $P\{Q\ge b\}\le P\left\{\sup_{t\in[0,T]}\left(Q(t)\right)>b\right\}$

$$Q(t) \cong \sup_{s \le t}\left(A(s,t) - C(t-s)\right) = \sup_{s \le t}\left(\sqrt{am}\left(B_\alpha(t) - B_\alpha(s)\right) - (C-m)(t-s)\right) \le$$

$$\le 2\sqrt{am}\left(\left|B_\alpha(t)\right|\right) - (C-m)t.$$

$$Q \cong \sup_{t \in [0,T]}\left(Q(t)\right) = \sup_{t \in [0,T]}\left(2\sqrt{am}\left(\left|B_\alpha(t)\right|\right) - (C-m)t\right) =$$

$$= 2\sqrt{am} \sup_{t \in [0,T]}\left(\left|B_\alpha(t)\right|\right) - T(C-m).$$

Then

$$P\{Q \ge b\} \le P\left\{\sup_{t \in [0,T]}\left(\left|B_\alpha(t)\right|\right) > \frac{b + T(C-m)}{2\sqrt{am}}\right\}$$

Put $x = \dfrac{b + T(C-m)}{2\sqrt{am}}$, the following theorem is fulfilled.

Theorem 16. For $x \ge D$ we have

$$P\left\{\sup_{t \in [0,T]}\left(\left|W_\alpha(t)\right|\right) > x\right\} \le 2\exp\left\{-\frac{(x-D)^2}{2A}\right\}$$

where

$$D = \sqrt{2}(a+b)$$

$$A = \frac{\gamma}{1-p} + \frac{p\beta^2}{(1-p)^2}$$

Simulation of FBM as input of some system with predetermined accuracy and reliability in the space $C([0,T])$ with respect of response of the system

Consider a time-invariant linear system with a real-valued square integrable impulse response function $H(\tau)$ which is defined on a finite domain $\tau \in [0,T]$. This means that the response of the system to an input signal $X(t)$ which is observed on $[-T,T]$ has the following form

$$Y(t) = \int_0^T H(\tau) X(t-\tau)\, d\tau, \quad t \in [0,T] \tag{14}$$

and $H \in L_2([0,T])$.

Some properties and estimators of impulse response function can be found in [31,33,36].

Suppose that the impulse response function is known. We also suggest that the input signal in system (14) is FBM with Hurst index α. From (14) follows that the response of the system (output) $Y(t)$ can be presented as

$$Y(t) = Y_\alpha(t) = \sum_{k=0}^{\infty}(\xi_k \cdot c_k(t) + \eta_k \cdot s_k(t)), \tag{15}$$

where the functions $c_k(\mathrm{t}),\ \mathrm{s}_k(\mathrm{t})$ equal

$$\mathrm{c}_k(\mathrm{t}) = b_k \int_0^T H(\tau)(1-\cos(\mathrm{y}_k(\mathrm{t}-\tau)))\,\mathrm{d}\,\tau,$$
$$s_k(\mathrm{t}) = a_k \int_0^T H(\tau)\sin(\mathrm{x}_k(\mathrm{t}-\tau))\,\mathrm{d}\,\tau. \qquad (16)$$

In this section we study the model construction of stochastic process $X_\alpha(\mathrm{t})$ and find the conditions that allows to approximate input signal $X_\alpha(\mathrm{t})$, taking into account the response of the system (output process) $Y(\mathrm{t})$ with given accuracy and reliability in Banach space $C([0,\mathrm{T}])$. To perform such simulation, we use the theory of Square-Gaussian random variables and processes. The similar results for Gaussian process were obtained in [32,35].

As a model of stochastic process $X_\alpha(\mathrm{t})$ we consider, as usual, a cutting off series in (1).

Definition 3. A stochastic process $X_{\alpha,N}(\mathrm{t})$ is called the model of the process $X_\alpha(\mathrm{t})$, if

$$X_{\alpha,N}(t) = X_N(t) = \sum_{k=0}^{N}(b_k\xi_k(1-\cos \mathrm{y}_k\ t) + a_k\eta_k \sin \mathrm{x}_k\ t).$$

If the model $X_N(\mathrm{t})$ is considered as an input signal of linear system then the output process is given in this way

$$Y_N(t) = \sum_{k=0}^{N}(\xi_k \cdot \mathrm{c}_k(t) + \eta_k \cdot \mathrm{s}_k(t)),$$

where the functions $c_k(\mathrm{t}), \mathrm{s}_k(\mathrm{t})$ are from (16).

Under $\xi_N(t)$ we denote the sum of square of the differences $X(\mathrm{t}) - X_N(\mathrm{t})$ and $Y(\mathrm{t}) - Y_N(\mathrm{t})$

$$\xi_N(t) = (X(\mathrm{t}) - X_N(\mathrm{t}))^2 + (Y(\mathrm{t}) - Y_N(\mathrm{t}))^2 \qquad (17)$$

Definition 4. We say that the model $X_N(\mathrm{t})$ approximates a stochastic process $X(\mathrm{t})$ taking into account the response of the system (14) with given reliability 1-ν, ν ϵ (0,1), and accuracy δ>0 in the space C([0,T]), if

$$P\left\{\sup_{t\in[0,T]} |\,\xi_N(t) - E\xi_N(t)\,| > \delta\right\} < \nu.$$

Square-Gaussian stochastic Processes

Now let's give the definitions and some properties of Square-Gaussian random variables and stochastic processes.

Assume that (T,ρ) is a compact metric space with metric ρ.

Definition 5. Let $\Xi = \{\xi_t, t \in T\}$ be a family of centered joint Gaussian random variables. A space $SG_\Xi(\Omega)$ is a space of Square-Gaussian random variables if any element $\eta \in SG_\Xi(\Omega)$ can be presented as

$$\eta = \zeta A \zeta^T - E\zeta A \zeta^T,$$

where $\zeta = (\xi_1, \xi_2, ..., \xi_n),\ \xi_k \in \Xi,\ k = \overline{1,n},\ A$ is a real-valued matrix or an element, $\eta \in SG_\Xi(\Omega)$ is a square mean limit of the sequence

$$\eta = \underset{n\to\infty}{\text{l.i.m}} (\zeta_n A \zeta_n{}^T - E\zeta_n A \zeta_n{}^T).$$

Remark 1.[26] The space $SG_\Xi(\Omega)$ is a Banach space with respect to the norm $\|\xi\| = \sqrt{E\xi^2}$.

Definition 5.A stochastic process $\xi(t), t \in [0,T]$, is Square-Gaussian if for any fixed t ϵ [0,T] a random variable $\xi(t)$ belongs to the space $SG_\Xi(\Omega)$ and $\sup_{t\in[0,T]} |\xi(t)| < \infty$.

We will use the following theorem on the tail distribution of the supremum of Square-Gaussian stochastic process. The proof of the theorem can be found in [26].

Theorem 17. [26] Assume that $\xi(t), t \in [0,T]$, is a separable Square-Gaussian stochastic process and

$$\sup_{|t-s|<h} \sqrt{D(\xi(t)-\xi(s))} \le \sigma(h) = kh^\alpha,\ \alpha \in (0,1], \tag{18}$$

where k is some constant. Then for x such that

$$x > \frac{2\sqrt{2}\max\{\delta_0, k(\mathrm{T}/2)^{\alpha}\}}{\alpha},$$

the inequality

$$P\left\{\sup_{t\in[0,T]} |\xi(t)| > x\right\} < 4e^{\frac{3}{\alpha}} \exp\left\{-\frac{x}{2\sqrt{2}\delta_0}\right\} \times \left(\frac{x\alpha}{2\sqrt{2}\delta_0}\right)^{2/\alpha} \left(1+\frac{2x}{\sqrt{2}\delta_0}\right)^{1/2}$$

holds true where $\delta_0 = \sup_{t\in[0,T]} (D(X(t)))^{1/2}$.

Without any difficulty it could be shown that zero-mean process $\xi_N(t) - E\xi_N(t)$ is Square-Gaussian, where $\xi_N(t)$ is from (17). So, Theorem 2 can be used in this case.

Denote

$$\phi_{kl}^1 = \phi_{kl}^1(\mathrm{t}) = \mathrm{b}_k\, a_l(1-\cos(\mathrm{y}_k\, t))(1-\cos(\mathrm{y}_l\, t)) + c_k(\mathrm{t})\,\mathrm{c}_l(\mathrm{t});$$
$$\phi_{kl}^2 = \phi_{kl}^2(\mathrm{t}) = 2(\mathrm{b}_k\, a_l(1-\cos(\mathrm{y}_k\, t))\sin(\mathrm{x}_l\, t) + c_k(\mathrm{t})\,\mathrm{s}_l(\mathrm{t})); \quad (19)$$
$$\phi_{kl}^3 = \phi_{kl}^3(\mathrm{t}) = a_k a_l \sin(\mathrm{x}_k\, t)\sin(\mathrm{x}_l\, t) + s_k(\mathrm{t})\,\mathrm{s}_l(\mathrm{t}).$$

Then by (14), (15) and (17) we have that the process $\xi_N(t)$ can be written in following form

$$\xi_N(\mathrm{t}) = \sum_{k=N+1}^{\infty}\sum_{l=N+1}^{\infty} (\phi_{kl}^1(\mathrm{t})\xi_k\xi_l + \phi_{kl}^2(\mathrm{t})\xi_k\eta_l + \phi_{kl}^3(\mathrm{t})\eta_k\eta_l). \quad (20)$$

Let us also denote the increments of the functions

$$\Delta\phi_{kl}^1 = \phi_{kl}^1(\mathrm{t}) - \phi_{kl}^1(\mathrm{s});\ \ \Delta\phi_{kl}^2 = \phi_{kl}^2(\mathrm{t}) - \phi_{kl}^2(\mathrm{s});\ \ \Delta\phi_{kl}^3 = \phi_{kl}^3(\mathrm{t}) - \phi_{kl}^3(\mathrm{s}). \quad (21)$$

To perform the main result, we first present the auxiliary relationships concerning mean, variance and variance of increments for the process $\xi_N(t)$.

Lemma 1. Let $\xi_N(t)$ be stochastic process from (17). Then

$$E\xi_N(\mathrm{t}) = \sum_{k=N+1}^{\infty} (\phi_{kk}^1(\mathrm{t}) + \phi_{kk}^3(\mathrm{t}));$$

$$D\xi_N(\mathrm{t}) = \sum_{k,l=N+1}^{\infty} \left(2(\phi_{kl}^1(\mathrm{t}))^2 + (\phi_{kl}^2(\mathrm{t}))^2 + 2(\phi_{kl}^3(\mathrm{t}))^2\right);$$

$$D(\xi_N(\mathrm{t}) - \xi_N(\mathrm{s})) = \sum_{k,l=N+1}^{\infty} \left(2(\Delta\phi_{kl}^1)^2 + (\Delta\phi_{kl}^2)^2 + 2(\Delta\phi_{kl}^3)^2\right).$$

Proof.

Since random variables $\xi_k, \eta_l, k \geq 0, l \geq 0,$ are jointly independent Gaussian with mean 0 and variance 1 then by (8) we have

$$E\xi_N(\mathrm{t}) = \sum_{k=N+1}^{\infty}\sum_{l=N+1}^{\infty}(\phi_{kl}^1(\mathrm{t})\,\mathrm{E}\,\xi_k\xi_l + \phi_{kl}^2(\mathrm{t})\,\mathrm{E}\,\xi_k\eta_l + \phi_{kl}^3(\mathrm{t})\,\mathrm{E}\,\eta_k\eta_l)$$

$$= \sum_{k=N+1}^{\infty}(\phi_{kk}^1(\mathrm{t}) + \phi_{kk}^3(\mathrm{t}))$$

To calculate the variance of the process $\xi_N(t)$, we should at first find the second moment

$$E(\xi_N(\mathrm{t}))^2 = E\left(\sum_{k,l=N+1}^{\infty}(\phi_{kl}^1(\mathrm{t})\,\mathrm{E}\,\xi_k\xi_l + \phi_{kl}^2(\mathrm{t})\,\mathrm{E}\,\xi_k\eta_l + \phi_{kl}^3(\mathrm{t})\,\mathrm{E}\,\eta_k\eta_l)\right)^2.$$

We use Isserlis formuls to compute the moment of the forth order for standard Gaussian random variables:

$$EX_1X_2X_3X_4 = EX_1X_2EX_3X_4 + EX_1X_3EX_2X_4 + EX_1X_4EX_2X_3.$$

Then we obtain

$$E(\xi_N(\mathrm{t}))^2 = E\sum_{k=N+1}^{\infty}\sum_{l=N+1}^{\infty}(\phi_{kk}^1(\mathrm{t})\phi_{ll}^1(\mathrm{t}) + 2(\phi_{kl}^1(\mathrm{t}))^2 + (\phi_{kl}^2(\mathrm{t}))^2 +$$
$$+\phi_{kk}^3(\mathrm{t})\phi_{ll}^3(\mathrm{t}) + 2(\phi_{kl}^3(\mathrm{t}))^2 + 2\phi_{kk}^1(\mathrm{t})\phi_{ll}^3(\mathrm{t})).$$

Therefore, the variance of stochastic process $\xi_N(t)$ equals

$$D\xi_N(\mathrm{t}) = E(\xi_N(\mathrm{t}))^2 - (\mathrm{E}\,\xi_N(\mathrm{t}))^2 =$$
$$= \sum_{k,l=N+1}^{\infty}\left(2(\phi_{kl}^1(\mathrm{t}))^2 + (\phi_{kl}^2(\mathrm{t}))^2 + 2(\phi_{kl}^3(\mathrm{t}))^2\right).$$

Similarly, it can be proved the formula for variance of process increments $\xi_N(\mathrm{t}) - \xi_N(\mathrm{s})$.

If we put $d_{kl} = \sup_{t\in[0,\mathrm{T}]}(2(\phi_{kl}^1(\mathrm{t}))^2 + (\phi_{kl}^2(\mathrm{t}))^2 + 2(\phi_{kl}^3(\mathrm{t}))^2)$. Then

$$\sqrt{D\xi_N(\mathrm{t})} \le \left(\sum_{k,l=N+1}^{\infty} d_{kl}\right)^{1/2} := \delta_0(\mathrm{N}).$$

Under some conditions it could be shown that

$$(D(\xi_N(\mathrm{t}) - \xi_N(\mathrm{s})))^{1/2} \le K(\mathrm{N})\cdot|\,\mathrm{t}-\mathrm{s}\,|^{\beta},\quad \beta\in(0,1], \tag{22}$$

where the function $K(\mathrm{N})$ depends on the quantity N.

The following theorem gives the conditions on the model to approximate FBM, taking into account the response of the system with predetermined accuracy and reliability.

Theorem 18. Suppose that the conditions (10) are satisfied. The model $X_N(\mathrm{t})$ approximates FBM $X_\alpha(\mathrm{t})$ with respect to the response (1) with reliability 1-ν, ν ϵ (0,1), and accuracy δ>0 in the space C([0,T]) , if for N The inequalities

$$\max\{\delta_0(\mathrm{N}), \mathrm{K(N)}\cdot(\mathrm{T}/2)^{\alpha}\} < \frac{\alpha\delta}{2\sqrt{2}},\ \alpha\in(0,1],$$

$$4e^{\frac{3}{\alpha}}\exp\left\{-\frac{\delta}{2\sqrt{2}\delta_0(\mathrm{N})}\right\}\times\left(\frac{\delta\alpha}{2\sqrt{2}\delta_0(\mathrm{N})}\right)^{2/\alpha}\left(1+\frac{2\delta}{\sqrt{2}\delta_0(\mathrm{N})}\right)^{1/2}<\nu,$$

hold true.

6. Conclusions

In the paper the important tasks were considered such as the analysis and development of the telecomunication networks, the estimation of Hurst index and the methods of statistical simulation of FBM. The obtained estimators of Hurst index is consistent and improve existing one. The great attention should be paid to Hurst index $\alpha = \frac{3}{4}$.

Methods for estimating the Hurst parameter are based on the Baxter sum and e Levi-Baxter limiting theorems. The confidence intervals of these etimators were found. A special cases of the second and the third orders were studied.

In the article, the simulation algorithms for FBM were giventhat use spectral representations. We also obtained the accuracy and reliability of the models.

In many applied problems, it is necessary to model the processes on the outputs of the linear systems. In this case, an algorithm for modeling such processes was investigated, given the accuracy and reliability.

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