Hilbert Mathematics Versus Gödel Mathematics. IV. the New Approach of Hilbert Mathematics Easily Resolving the Most Difficult Problems of Gödel Mathematics

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Abstract. The paper continues the consideration of Hilbert mathematics to mathematics itself as an additional "dimension" allowing for the most difficult and fundamental problems to be attacked in a new general and universal way shareable between all of them. That dimension consists in the parameter of the "distance between finiteness and infinity", particularly able to interpret standard mathematics as a particular case, the basis of which are arithmetic, set theory and propositional logic: that is as a special "flat" case of Hilbert mathematics. The following four essential problems are considered for the idea to be elucidated: Fermat's last theorem proved by Andrew Wiles; Poincaré's conjecture proved by Grigori Perelman and the only resolved from the seven Millennium problems offered by the Clay Mathematics Institute (CMI); the four-color theorem proved "machine-likely" by enumerating all cases and the crucial software assistance; the Yang-Mills existence and mass gap problem also suggested by CMI and yet unresolved. They are intentionally chosen to belong to quite different mathematical areas (number theory, topology, mathematical physics) just to demonstrate the power of the approach able to unite and even unify them from the viewpoint of Hilbert mathematics. Also, specific ideas relevant to each of them are considered. Fermat's last theorem is shown as a Gödel insoluble statement by means of Yablo's paradox. Thus, Wiles's proof as a corollary from the modularity theorem and thus needing both arithmetic and set theory involves necessarily an inverse Grothendieck universe. On the contrary, its proof in "Fermat arithmetic" introduced by "epoché to infinity" (following the pattern of Husserl's original "epoché to reality") can be suggested by Hilbert arithmetic relevant to Hilbert mathematics, the mediation of which can be removed in the final analysis as a "Wittgenstein ladder". Poincaré's conjecture can be reinterpreted physically by Minkowski space and thus reduced to the "nonstandard homeomorphism" of a bit of information mathematically. Perelman's proof can be accordingly reinterpreted. However, it is valid in Gödel (or Gödelian) mathematics, but not in Hilbert mathematics in general, where the question of whether it holds remains open. The four-color theorem can be also deduced from the nonstandard homeomorphism at issue, but the available proof by enumerating a finite set of all possible cases is more general and relevant to Hilbert mathematics as well, therefore being an indirect argument in favor of the validity of Poincaré's conjecture in Hilbert mathematics. The Yang-Mills existence and mass gap problem furthermore suggests the most general viewpoint to the relation of Hilbert and Gödel mathematics justifying the qubit Hilbert space as the dual counterpart of Hilbert arithmetic in a narrow sense, in turn being inferable from Hilbert arithmetic in a wide sense. The conjecture that many if not almost all great problems in contemporary mathematics rely on (or at least relate to) the Gödel incompleteness is suggested. It implies that Hilbert mathematics is the natural medium for their discussion or eventual solutions.

Keywords: Fermat's last theorem (FLT), four-color theorem, Gödel mathematics, Hilbert arithmetic, Hilbert mathematics, Perelman's proof, Poincaré's conjecture, qubit Hilbert space, quantum information, Wiles's proof, Yang-Mills existence and mass gap problem

I INSTEAD OF INTRODUCTION: GÖDEL MATHEMATICS VERSUS HILBERT MATHEMATICS, OR GÖDELIAN MATHEMATICS AMONG HILBERT MATHEMATICS

More or less jokingly, one can offer "Lev Tolstoy's heuristics" extracted from the famous first sentence of "Anna Karenina" about 'happy and unhappy families' only substituting them by "resolved and unresolved problems" therefore hinting at the idea that if one manages to unify the unresolved problems, they are already likened to the "happy" resolved ones therefore implicitly giving notice about their forthcoming solutions.

The paper intends to demonstrate that general method to a few famous problems and their solutions if those are suggested and established already (respectively to their possible eventual future solutions if those do not exist yet). In other words, or following the metaphor about "Tolstoy's heuristics", their solution is much easily to be a general and single one "uniting in their own unhappiness" than be alienated and concentrated only on the personal "grief" being thus absolutely divided from each other within the particular insolvability. A "psychological support group" for them may help, and the present paper can be seen from that metaphorical viewpoint as its "organization" similar (for example) to meetings of "Alcoholics Anonymous". Indeed, the unhappiness of each of them is individual, but nonetheless their problem is common and uniting and meeting in a joint group, they approach its solution.

Four famous mathematical problems are chosen rather intentionally to be able to demonstrate different specific, but typical features of their insolvability, paradoxically taking steps to shared solvability impossible being alienated "each alone in the own unhappiness". Those problems are: (1) Fermat's last theorem (accompanied by Andrew Wiles's solution)[1]; (2) Poincaré's conjecture[2] (together with Grigori Perelman's solution[3] and being the only resolved "CMI Millennium problem"); (3) the four-color theorem[4] (supplied with a computer solution alone); (4) the Yang-Mills mass gap problem[5] (one of the rest six unresolved "CMI Millennium problems"). They seem to be quite different in their nature or affiliation to various enough mathematical branches in order to be able to demonstrate the fruitfulness of their unification (at first glance so obviously unlike to each other).

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[1] Besides Wiles's original paper (1995), there are enough comments on his result (e.g., Saitō 2015; 2013; Zhang 1998; Chiaho 1996), including those (e.g. Edwards 1975) about the partial result of Kummer (1847) as fat as they are relevant to the context of the present paper. The paper of Veleman (1997) discusses the link of Fermat's last theorem with Hilbert's program.

[2] For example, Maia (2011); Morgan (2009); Morgan, Tian 2007; O'Shea (2007; 2007a); Strzelecki, Shenitzer 2006; Morgan (2004); Kreck (2001); Gillman (1990); McMillan, Thickstun (1980); Gross (1969); Papakyriakopoulos (1962).

[3] Perelman 2002; 2003; 2003a.

[4] There exists a series of papers (e.g., Secco, Pereira 2017; Cooper, Rowland, Zeilberger 2012; Ohnishi 2009; Eliahou 1999; Eliahou, Lecouvey 1999; Fritsch, Fritsch 1998; Burger, Morgan 1997; Bar-Natan 1997; Heller 1997; Modrak 1989; Themaat 1989; Appel 1984; Detlefsen, Luker 1980; Bernhart 1977) discussing the four-color theorem in a rather methodological context relevant to philosophy of mathematics and thus here.

[5] For example, Dynin (2014) or Jaffe, Witten (2006).

The "lost solution of Fermat's last theorem" (claimed by himself) is an essential part of its legend challenging generations of mathematicians during almost four centuries, furthermore featured by an elementary formulation. Wiles's solution as a corollary from the modularity theorem linking the discrete modular forms with continuous elliptic curves seems to be absolutely inaccessible to Fermat and his age, in which the corresponding mathematical concepts and ideas did not exist thoroughly. The criticism to Wiles's proof emphasizes that a "Grothendieck universe" is involved though implicitly as a necessary condition for it thus transcending the standard set theory (e.g., in its ZFC version).

In other words, the blame can be reformulated as the statement that Fermat's last theorem is unprovable in ZFC set theory, Peano arithmetic, and propositional logic, for example by virtue of the suggestion that it is a Gödel insoluble statement in the exact meaning of his "incompleteness paper" (1931). Even more: one can prove quite rigorously that it is actually such a statement by demonstrating that it satisfies Yablo's paradox implying for it to be a Gödel insoluble statement (Penchev 2021 March 9). Then, Wiles's proof being correct cannot but be beyond the standard set theory (i.e., in any version equivalent to its ZFC version).

One can visualize that transcendency of Wiles's proof quite elementary, by virtue that it is a corollary from the modularity theorem linking the modular forms, originating from Peano arithmetic, and elliptic curves, which being continuous needs set theory: if it is in its ZFC version (rather than as a Grothendieck universe), the connection meant by the modularity theorem cannot but meet the Gödel dichotomy of the relation of (Peano) arithmetic and (ZFC) set theory: either incompleteness[6] or contradiction. So, after one has rejected the alternative of "contradiction" after granting the correctness of the proof, the prevention of "incompleteness" implies for it to transcend the ZFC set theory into some Grothendieck spaciousness though implicitly.

Indeed, the inaccessible cardinals meant by any Grothendieck universe are in an inverse form since the continuous elliptic curves necessary for the modularity theorem do not need more powerful cardinals than that of continuum (avoiding even Cantor's continuum hypothesis). However, those inversely inaccessible cardinals appear inevitably due to the necessity to be "filled the gap" between the inherently finite Peano arithmetic and the actually infinite (after Cantor) set theory, for example by the following construction inverse to the original Grothendieck axiom that the power set (i.e., the set of all subsets) of any set belongs also to any "universe" claiming to be Grothendieck:

To avoid either alternative of the continuum hypothesis, one starts from a countable set, but "backwards": that is by postulating that there always exists a set such that its power set is a set given in advance. Then repeating again and again the so-defined set theoretical operation (being always possible), one cannot reach any finite set by it in any way since the power set of a finite set

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[6] Enough papers consider the concept of incompleteness in a context both philosophical and mathematical (Kennedy 2022; Plato 2020; Smullyan 1992), including in relation to Hilbert's program (Detlefsen 1990) as well as to that of completeness (e.g., Dean 2020; McAloon 1978). The concept of the alleged incompleteness of quantum mechanics (e.g., Garola 1993; 1992), started by Einstein, Podolsky, and Rosen's famous paper (1935) are usually meant not to be connected with the Gödel incompleteness (e.g., Held 2015; Harrigan, Spekkens 2010; Scherer, Busch 1993).

is again finite therefore constructing an inverse Grothendieck universe in the vein attempt for the *set* of all natural numbers (after set theory generalized to a Grothendieck universe) to be reduced to all natural numbers (being inherently finite for the axiom of induction, e.g., in Peano arithmetic). In fact, the nature of that vein attempt, resulting in an inverse Grothendieck universe in the final analysis, is the same as the alternative of incompleteness in the Gödel dichotomy about the relation of arithmetic to set theory. Consequently, Wiles's proof needs a Grothendieck universe (though in an inverse form) for the modularity theorem linking an arithmetic structure such as the class of modularity forms to a continuous structure such as the class of elliptic curves.

However, one can avoid any Grothendieck universe in a quite different way, only returning back to the "innocence" of Fermat's age not consumed yet the "original sin" of Cantor's actual infinity. So, the distinction between the finiteness of arithmetic and the actual infinity of set theory had not appeared up to then. Anyway, one can restore that "paradise" from our contemporary viewpoint by means of a Husserlian "epoché to infinity" (similar to his original "epoché to reality"[7]) only avoiding any unambiguous answer of the question about any mathematical entity whether it is finite or infinite. Indeed, nobody including Fermat himself in his age did not question whether natural numbers are finite or infinite at least in that absolutely rigorous formulation opposing finiteness for the axiom of induction in Peano arithmetic to actual infinity due to the axiom of infinity in ZFC set theory.

So, the Husserlian "epoché to infinity" (in fact, following the formal structure of the original epoché to reality) allows to prevent involving any "inverse Grothendieck universe" (in the exact meaning as above) in relation to the eventual proof of Fermat's last theorem. One may suggest "Fermat arithmetic", naturally corresponding to that "epoché to infinity" just because the opposition of finiteness (after the axiom of induction) and Cantor's "actual infinity" had not been yet articulated in his epoch. The contemporary question about all natural numbers versus the set of all natural numbers obeyed the epoché at issue.

However, the present introductory section restricts itself only to emphasize that the distinction between Fermat's last theorem in the context of his own time and itself, but in the contemporary context featured by the aforementioned Gödel dichotomy about the opposition of arithmetic and set theory can be formally represented by an additional dimension, to which the "innocence" of Fermat arithmetic can be anyway restored by that "epoché to infinity" and sharing the same structure as a bit of information; that is: Fermat arithmetic in "Eden before the original sin" of Cantor's actual infinity (for the state of a bit *before* the choice of either alternative) versus the contemporary opposition of Peano arithmetic and set theory (for the state of a bit *after* the choice of either alternative).

Indeed, the addition of one more dimension can be interpreted furthermore philosophically and ontologically, in fact, following Husserl's original epoché to reality and the Cartesian organization of cognition in Modernity, after which mathematics is restricted only within Descartes's "mind"

[7] There are enough papers (Hewitson 2014; Overgaard 2010; Lübcke 1999; Pentzopoulou-Valalas 1988; Lenkowski 1978; Küng 1975; Bossert 1974; Löwit 1957), discussing Husserl's epoché as a skeptic attitude to whatever and thus particularly relatable to the distinction of finiteness versus infinity.

versus "body", in turn an inherent subject of physics. Accordingly, one can admit "phenomenological mathematics" in a Husserlian manner (though he saw logic in a rather Aristotelian way as what "phenomenological mathematics" is).

Next, one can divide Gödel mathematics from Hilbert mathematics[8] according to the interpretation of the Gödel incompleteness statement[9] (1931): either a theorem or an axiom justified in detail in the first part of the paper (Penchev 2022 October 21). Furthermore, one can distinguish Gödelian mathematics featured by the "zero distance of infinity from finiteness" among the class of all various Hilbert mathematics and corresponding unambiguously to Gödel mathematics. Then, the new dimension able to distinguish the original and contemporary contexts of the interpretation of the same formulation of Fermat's last theorem as above can be identified as the additional dimension meant by the parameter of the distance between finiteness and infinity, being inherent to the distinction of Hilbert mathematics to Gödel mathematics.

One can continue that approach further, to the interpretation of the next enumerated fundamental mathematical problem: Poincaré's conjecture and its recognized solution suggested by Gregory Perelman. Poincaré's conjecture states the topological equivalence (i.e., the existence of a homeomorphism) of the usual three-dimensional Euclidean space and the four-dimensional unit 3-sphere therefore involving a mapping between a three-dimensional topological structure such as Euclidean space and a four-dimensional topological structure such as a unit 3-sphere. In other words, the addition of a new dimension distinguishes the sides of the investigated equivalence though only in a topological meaning after which they are interpreted only as topological rather than vector spaces.

One may unfold the four-dimensional unit 3-sphere into three dimensions and then apply a relevant homeomorphism, after which the unfolding at issue would be isomorphic to either domain of Minkowski space (whether real or imaginary). Then, Poincaré's conjecture acquires immediately a physical interpretation and sense since the imaginary domain of Minkowski space implies special relativity, on the one hand, and it can be opposed to its real domain just as the pair of locality and nonlocality[10], but on the other hand, this means the propagation of any

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[8] Hilbert mathematics can be considered as a generalized realization of Hilbert's program (e.g. Doherty 2019; Sieg 2013; 1988; Stenlund 2012; Akiyoshi 2009; Franks 2009; Feferman 2008; 1998; Makowsky 2008; Rathjen 2005; Shapiro, 2005; Raatikainen 2003; Zach 2003; Niebergall, Matthias 2002; Corry 1997a; Veleman 1997; Webb 1997; Blanchette 1996; Demopoulos 1994; Gauthier 1994; Ignjatović 1994; Kaye 1993; Detlefsen 1990; 1986; Simpson 1988) furthermore relying on the separable complex Hilbert space utilized by quantum mechanics (e.g. Lacki 2000; Raviculé, Casas, Plastino 1997; Rédei 1996; Gudder 1983) as a tool able to unify arithmetic, geometry and even physics (e.g., Sauer, Majer, eds. 2009; Tresoldi 2009 Brading, Ryckman 2008; Majer 2006; Majer Sauer 2006; Schirrmacher 2003; Shima, 2002; Rowe 2001; Arnaudon, Paycha 1997; Corry 2004; 1998; 1997) in the foundations of mathematics and then, its three "whales": arithmetic, set theory and propositional logic (e.g. Hilbert, Bernays 2013; Hilbert 1905). Hilbert mathematics as here suggests Pythagoreanism, which corresponds more or less to Hilbert's implicit philosophy (e.g., da Silva 2016; 2000; Majer, Sauer 2006; Gillies 1999; Mancosu 1999; Majer 1997; Gauthier 1994; Giaquinto 1983; Mahnke 1977).

[9] The statements are literally two different in Gödel's paper: "Satz VI" and "Satz X". Anyway, they can be unified from the viewpoint of Hilbert mathematics or by a kind of "epoché" to their difference.

[10] In detail in: *Penchev 2023 March 13.*

electromagnetic wave in a vacuum, the usual three-dimensional Euclidean space, and the relevance of the representation of the latter physical process by the former mathematical model:

Indeed, if they are not topologically equivalent, respectively homeomorphic to each other, this would reflect into some violation of the representation of causality, which would be available at any topological heterogeneity in either of the mathematical model and physical process. Nonetheless, the topological heterogeneity at the light cone and the bound of the speed of light in a vacuum seems to be obvious, implying for Poincaré's conjecture to be a paradoxical and even false statement, at least at first glance. In fact, special relativity deduces that topological heterogeneity by metrical considerations about the unattainability of the speed of light in a vacuum for whatever body with any nonzero mass at rest or that of any speed exceeding it, thus cutting the real domain of Minkowski space as physically nonsense. So, the physical interpretation of Poincaré's conjecture by means of special relativity suggests some mismatch between them: whether as the falsity of the former or as the incompleteness of the latter.

However, if one adds the physical realm of entanglement and nonlocality to the standard physics of locality within the light cone or the imaginary domain of Minkowski space, the disparity to Poincaré's conjecture vanishes in thin air. In other words, the physical description is incomplete which Einstein himself saw very well and for which he created general relativity without touching the postulate of locality, empiricism, and universal and absolute experimental repeatability of any phenomenon claiming to be physical. On the contrary, quantum mechanics rejected that postulate and met Einstein's mockery by the pejorative epithet of that *spooky* action at a distance or his sardonic metaphor of *God playing dice*, both really corresponding to the nonlocal and probabilistic approach of quantum mechanics.

Indeed, the 2022 Nobel Prize for entanglement and quantum information decided the scientific "trial" lasted about a century of "Albert Einstein against Niels Bohr" in favor of the latter formally. However, that decision turns out to be rather "Solomonic" since Einstein's local position was not rejected especially after the complement of general relativity, but only verified to be equivalent to nonlocality or the completeness of quantum mechanics advocated by Bohr. One can immediately interpret it by Poincaré's conjecture: the usual three-dimensional Euclidean space being inherently boundless is therefore homeomorphic to the four-dimensional unit (and thus bounded) 3-sphere which in turn is homeomorphic to all Minkowski space and to both imaginary and real domains of it.

One can notice that Poincare's conjecture (as Fermat's last theorem) shares the formal structure of a bit of information where Euclidean space corresponds to the state "before the choice of either alternative", each of both is accordingly either the real area or the imaginary area of Minkowski space in the "state after choice" meant by the definition of a bit of information. In other words, one may offer the following generalization of Poincare's conjecture in relation to a bit of information: the state before choice is homeomorphic to the state after choice regardless of that additional dimension or bound appearing in the latter case to the former. Indeed, that topological equivalence seems to be rather counterintuitive as well as its physical interpretation by causality elucidated above in relation to the particular case of Poincaré's conjecture: the choice meant by a bit of information does not violate causality.

If one manages to demonstrate the "homeomorphism of a bit of information" in the above sense, this would imply Poincare's conjecture, and the homeomorphism at issue can be also embedded in the relation of Hilbert mathematics to Gödel mathematics: e.g., as follows. One can oppose the former to the later distinguishing them as the two formal and logical alternatives of a single axiom (similar to the Fifth postulate in relation to the pair of Euclidean and non-Euclidean geometry) though being a metamathematical one: whether the Gödel incompleteness statement is an axiom or a theorem (a problem discussed in the first part of the paper: *Penchev 2023 May 3; 2023 January 3; 2022 October 21*).

Indeed, the structure of a bit of information can be immediately juxtaposed to that metamathematical axiom: any bit consists of two oppositions though complementary to each other. The one of them is explicit, and a bit is often identified only with it: that between the two alternatives of it, for example such as either "0" or "1 which can be recorded in a Turing machine tape cell. The other one is that between the state before any choice versus that of either choice, or in the example, an empty cell before recording versus it after either value has been already recorded.

Then, the idea of "Fermat arithmetic" mentioned above may illustrate the state before the forced choice between "either incompleteness or contradiction" in the Gödel dichotomy about the relation of arithmetic to set theory just due to the "epoché to infinity" inherent to Fermat arithmetic by virtue of its "innocence". However, it can be again restored also after the "original sin" of Cantor's actual infinity has been already "consumed" following formally the structure of Riemann's approach to the pair of Euclidean and non-Euclidean geometries by the parameter of space curvature newly introduced by him:

Its analogue as to the pair of Hilbert and Gödel mathematics is the distance between finiteness and infinity, after which one can distinguish Gödelian mathematics in the framework of the former versus Gödel mathematics itself, i.e. opposed to the class of all Hilbert mathematics by the axiom at issue in a sense analogical to that after which one can state that Euclidean geometry is featured by a "zero" space curvature though that proposition does not make sense to the proper (or original) geometry of Euclid, since it makes sense only in the context of the class of all Riemannian geometries. In other words, a zero distance of infinity to finiteness is assigned to Gödelian mathematics rather than to Gödel mathematics, to which it does not make sense in a similar way.

Following the just sketched interpretation of a bit of information by the pair of Hilbert and Gödel mathematics, the problem about the homeomorphism of a bit of information (also underlying Poincaré's conjecture and implying it in the final analysis) can be represented as follows: the parameter of the distance between finiteness and infinity is a continuous variable; nonetheless, it is topologically equivalent (i.e. homeomorphic) to the discrete logical opposition of Hilbert mathematics and Gödel mathematics as the true and false alternatives of a single proposition, which is the aforementioned metamathematical axiom in the case at issue.

So, the homeomorphism of a bit of information (and hence, Poincaré's conjecture itself) is demonstrated as far as the above double construction linking Hilbert mathematics and Gödel mathematics interpreted that homeomorphism. One can generalize that the addition of a new dimension is always homeomorphic to the case where that dimension is not yet added, on the one hand and topologically, but on the other hand two mathematical structure described by "n" axioms and by the same "n" axioms, to which is added one more axiom consistent to the previous "n" ones, are homeomorphic, and that generalization implies Poincaré's conjecture. For example, one can build models of the "n+1" structure into the "n" structure, a necessary condition for which is to be topologically equivalent.

One can reduce by virtue of the axiom of induction any structure consistently represented by any "n" axioms to a single bit of information in a homeomorphic way, which allows for elucidating why the structure of a bit of information is so fundamental for mathematics: *all mathematics structures as long as being consistently formulated by "n" axioms are homeomorphic to a bit of information*.

One can pass further to the four-color theorem in the context of Hilbert mathematics and its opposition to Gödel mathematics, which is the standard one. In fact, the last result (in italic) can be interpreted by that theorem as follows: any consistent system consisting of "n" axioms can be represented as a two dimensional "map" of sets by Venn diagrams, so that any set differentiated by virtue of the axiomatics at issue is a "country" possessing unambiguous boundaries due to the consistency. Then, the cited result just above confirms that any axiomatics represented by "n" consistent axioms is topologically equivalent to the two opposition of a bit of information, which can furthermore be visualized by the "four colors" sufficient for the Venn diagram map able to interpret any consistent axiomatics.

The unambiguous correspondence of any two dimensional "map" to a certain "n" axiomatics by the mediation of Venn diagrams for any relevant set seems to be obvious as well as that of the two oppositions of a bit of information by "four colors". So, only the "coloring" (of any two-dimensional map by "four colors") to be equivalent to homeomorphism is more or less problematic and needs additional considerations. Those will be discussed in detail in *Section VII*, but one can give simple intuitive or logical tenets in its favor:

The coloring of an area in the same color means just that any subarea is homeomorphic to the entire area, and the uniformity of a single color enough for it serves to notate just that circumstance. If the coloring reaches any discontinuity preventing that homeomorphism, this is a "boundary" distinguishing the former domain from that one starting at the boundary and thus need any other color (i.e., not to be the same as in the former domain). So, the homeomorphism needs the same color for continuity, but any other color after the boundary of the discontinuity, and the coloring notation is able to represent it unambiguously.

The other intuitive tenet consists in the projection of any two-dimensional map onto both abscissa and ordinate of any two-dimensional orthogonal coordinate system associable with the plane of the map. Obviously, two colors are sufficient for each of both axes, and since they are absolutely enough to describe metrically unambiguously any "country" on the map, these four topological colors are to be sufficient for the topological description of the map.

One may notice that the four-color theorem relevant to the standard, i.e., Gödel mathematics should be related to the "flat" case of Gödelian mathematics in the framework of Hilbert mathematics. On the contrary, any "nonzero" finite distance between finiteness and infinity implies the necessity of one more, i.e., "fifth" color to be involved whether for "no man's areas" of intuitionistic mathematics or for the "condominium areas" of "dialectic mathematics", thus the four-color theorem being invalid in either of them.

Accordingly, that fifth color can be likened to "that of physics" or more precisely, that of quantum information or action after being enumerated, accordingly, to the two colors of the one axis or to the two other colors of the other axis. Following the distinction of Emmy Noether's theorems (1918), either of both axes can be called the "conservation one" and be opposed to the Lie-group one for the other axis. Meaning the last consideration, one can admit that Poincaré's conjecture would not be also and analogically valid in Hilbert mathematics in general, but only in Gödel (or Gödelian) mathematics.

The last problem mentioned above is the Yang-Mills mass gap yet unresolved and one of the seven CMI Millennium problems[11] as Poincaré's conjecture, which is however resolved already by Gregory Perelman. It states that for any usual (i.e., compact and simple) gauge group exists a relevant Yang-Mills theory on the four-dimensional Euclidean space so that a Higgs mechanism corresponds and the lightest particle possesses a certain finite mass at rest, which is the "mass gap" at issue. This is its standard formulation rather sounding as a special physical problem than as a fundamental mathematical one and even still less suggesting any relation to the foundations of mathematics being the proper subject of the present paper by the opposition of Hilbert mathematics and Gödel mathematics so that the only hinted link needs an elucidation in detail:

Abandoning the redundantly precise formulation, that mass gap problem means the following. Any gauge symmetry (thus relevant to the "gauge group" mentioned above) equates local and global structures as being just symmetric to each other, i.e., identical in a class of equivalence defined by that symmetry. For example, the Standard model, also involving a relevant gauge theory (in fact, the Yang-Mills mass gap problem is a mathematical generalization of the mathematical formalism *ad hoc* working very well in the Standard model, but *inexplicably why*), equates the global and local separable complex Hilbert spaces.

In other words, any gauge theory can be understood as a generalization of the equivalence of both "languages" of classical mechanics: "Lagrangian" and "Hamiltonian" (languages). It means that the class of all local infinitesimal neighborhoods of any variable can be considered as a second variable absolutely independent of the former (i.e., translating from "Lagrangian" language into "Hamiltonian") as well as vice versa (i.e., from "Hamiltonian" language into "Lagrangian"). The general case of symmetry can mean any bijection (and even any mapping so that the reverse mapping not to be a function) rather than an identity as the simplest example of the translation

[11] The paper edited by Carlson, Jaffe, Wiles (2006) is about the Millennium Prize problems in detail.

between "Hamiltonian" and "Lagrangian" languages, but absolutely enough for elucidating the relation of the mass gap problem to the foundations of mathematics.

So, if that gauge symmetry is given, a Yang-Mills theory on the four-dimensional Euclidean space exists in turn implying an analogue of the spontaneous violation of symmetry (in the particular case of the Standard model represented by the "Higgs mechanism") so that a minimal finite mass at rest corresponds (i.e., a "mass gap" to the zero mass). Indeed, the four-dimensional Euclidean space implies the unit 3-sphere meant by Poincaré's conjecture and its two dual unfoldings together constituting Minkowski space, on the one hand, and furthermore homeomorphic to the two dual qubit spaces corresponding to the two dual separable complex spaces of classical quantum mechanics, on the other hand.

Then and as to the Standard model, the Yang-Mills theory (in a narrow and proper sense) means the split of the single electro-weak interaction into both electromagnetic and weak interactions and accordingly: the united or "entangled" symmetry "$\{[U(1)] \otimes [SU(2)]\}$ into the tensor product of them (notated by the missing brackets "$\{\}$"): $[U(1)] \otimes [SU(2)]$. Furthermore, the latter case implies the mass gap at issue by virtue of the Higgs mechanism. So, one can admit that an analogical link between a Yang-Mills theory and the generation of a relevant mass gap can be restored also in the generalized case seen by the proper Yang-Mills mass gap problem.

So, the crucial difficultness of the problem can be rather concentrated onto the *existence* of a relevant Yang-Mills theory for any gauge theory or particularly, onto the way, by which Yang-Mills theory in a narrow and proper sense follows from the gauge theory relevant to the Standard model: a link absolutely successful, but added *ad hoc* and yet unexplained *why*. In fact, just this is the core of the relation of the mass gap to the foundations of mathematics.

Then, one can notice the way in which the problem of the existence of a certain Yang-Mills theory is relevant to the foundations of mathematics in the context of both Hilbert and Gödel mathematics and the distinguishing parameter of the distance between finiteness and infinity. Indeed, finiteness and infinity, speaking loosely, are "gauge symmetric" and thus constitute a rather elementary, and thus simple and compact gauge group therefore satisfying the conditions of the Yang-Mills existence and mass gap problem. If one considers the space of *infinity* in an intuitive sense, any point of it is supplied by an infinitesimally small neighborhood of *finiteness* furthermore identical for all points of the space of infinity.

So, the constitution of a gauge group of infinity starting from the distinguished finiteness and infinite is no other than a serial reinterpretation of the translation from "Hamiltonian" to "Lagrangian" (languages), and the Gödel dichotomy about the relation of arithmetic (for "finiteness") to set theory (for "infinity") can be immediately reformulated into gauge-group terms as a rather trivial property originating from both definition and opposition of local and global spaces needing the local space to be doubled, by which a relevant "Yang-Mills theory of finiteness and infinity" is already suggested though implicitly.

Then, the qubit Hilbert space (just as a reinterpretation of the usual separable complex Hilbert space of classical quantum mechanics into the four-dimensional Euclidean space featuring Yang-Mills theories) as a complementary counterpart of Hilbert arithmetic in a narrow sense can be considered to be that "Yang-Mills theory of finiteness and infinity", after the emancipation of the U(1) group for any single "empty" qubit and the SU(2) group for the pair, in which it is accompanied by its conjugate counterpart, makes sense and really takes place.

In other words, the "Yang-Mills theory of finiteness and infinity" tends to describe the decoherence of any empty qubit into the conjugate pair of both complementary qubits, into which the Higgs mechanism is able to feature a certain point as a "spontaneous violation of symmetry", after which the absolute separation of U(1) from SU(2) is ultimately accomplished, and a physical entity with a certain energy (and changeable in the course of time) appears "ex nihilo" or more precisely, from the physically dimensionless free variable of quantum information consisting only of empty qubits in fact identical to single one. This is the "Miracle of the Creation", however described rigorously and scientifically.

At least at an intuitive level (just sketched above), Hilbert mathematics states that a relevant Yang-Mills theory always exists in its framework to any simple compact gauge group exemplified by the aforementioned "gauge group of finiteness and infinity" since Hilbert mathematics itself is the class of all possible Yang-Mills theories of that kind. Meaning that circumstance, the second "half" of the problem (namely whether a mass gap is necessary as it follows from the corresponding Yang-Mills theory) can be unambiguously resolved, though.

In fact, the parameter of the distance between finiteness and infinity corresponds to the mass gap at issue. So, whether it exists or not is a matter of a convention or postulate distinguishing Gödel mathematics from Hilbert mathematics, on the one hand, and the "flat" or "zero" case of Gödelian mathematics among the latter, on the other hand. In fact, the mass gap problem is extracted from a physical theory in the framework of the Standard model and generalized to be mathematical and even heralded by CMI as one of the "seven Millennium problems".

So, if one grants for it to be a physical problem, he or she has granted in advance that the "flat" Gödelian mathematics is not the case. However, if it is presupposed to be mathematical (as CMI does) just Gödel or Gödelian mathematics is the case in default, and the mass gap problem should be resolved negatively however only by virtue of its conventional context consisting in the random fact that it is formulated by CMI (rather than for example by CERN, after which and its context the same problem should be resolved positively).

However, if one continues the same course of thought back to the first "half" of the problem about the existence of a relevant Yang-Mills theory under the occasional condition to be established by CMI (rather than by CERN), the solution can be also interpreted to be unambiguous since it is positive in Gödelian mathematics (because it is a particular case of Hilbert mathematics) but negative in Gödel mathematics (because it is logically opposed to Hilbert mathematics) though Gödel and Gödelian mathematics by themselves are homomorphic to each other and distinguishable from each other only by virtue of their different context. If one tries to specify whether Gödelian or Gödel mathematics is meant by the referred axiomatic frameworks[12] of what a Yang-Mills theory is, the ambiguity cannot be removed since that axiomatics does not articulate

[12] In more detail in the previous parts: *Penchev 2023 May 3; 2023 January 3; 2022 October 21*.

the distinction between Gödel and Gödelian mathematics being a proper subject only in the present paper (or in one of its precedent papers: *Penchev 2022 October 21*).

So, the "legal case" of the "Yang-Mills existence and mass gap problem" turns out to be quite vague, being dependent on its context: whether physical or mathematical, and if the context is granted to be mathematical whether that of Gödel mathematics or within the framework of Hilbert mathematics, and if the latter is the case: whether Gödelian mathematics or non-Gödelian mathematics.

However, the precedent for a theorem to be true in an axiomatic system, but false in another being meaningful in both appeared since Lobachevski's age: the historically first case of two alternative axiomatic systems on the same subject, which differ from each other only in the Fifth postulate of Euclid: for example, the theorem about the sum of the angles of a triangle being equal to "2π" in Euclidean geometry and a variable depending on Riemann's space curvature in non-Euclidean geometry ("$> 2\pi$" in the hyperbolic geometry of Lobachevsky in particular).

One might use the metaphor of a real legal case, which can be decided differently according to two or more different national legislations, which is the reason for international contracts to be explicitly included a special clause about the legislation of which state to be meant if a litigation between the parties of the contract at issue appears in the future. However, the "Yang-Mills existence and mass gap problem" is anyway special as far as it means a metamathematical axiom differing Hilbert mathematics from Gödel mathematics, which can be interpreted as a mathematical axiom in the framework of the former, distinguishing Gödelian mathematics from non-Gödelian mathematics.

Continuing the legal metaphor above, one is to suggest a too complicated case where the contract at issue is inadmissible according to the constitution of some country (for "Gödel mathematics"), but possible according to that of another country, in the legislation of which is furthermore provided a special relation to counterparts belonging to states, the constitutions of which do not allow for that kind of contracts. The "Yang-Mills existence and mass gap problem", following that metaphor, would represent the most difficult special legal case where the one of the counterparties of contract belongs to a country, the constitution of which does not admit the contract at issue at all. This complicated "legal" situation is illustrated above by different solutions of the same problem in dependence on the context of its establishment: whether by CMI (the real case) or by CERN (a hypothetical and counterfactual case).

The reason for that exceptional "legal" complicatedness in question is the fact that its solution depends on its ontological status: whether Cartesian or Pythagorean; originating furthermore of its extraordinary way to be formulated, namely, as a mathematical generalization of a real and very successful physical theory (the Yang-Mills one) in the framework of the Standard model. So, if one grants that physics and mathematics are divided on the two opposite "shores" of the Cartesian "abyss", the solution is one, but if one unifies them in a Pythagorean manner, the solution turns out to be quite different.

Nonetheless, the problem of whether our universe is "Cartesian" or "Pythagorean" is absolutely reasonable. The irrefutable experimental facts of dark mass and dark energy fit very well to the case of our universe to be "Pythagorean", where a "dark phase" of it is opposed to its "light phase" identifiable furthermore with all claiming to be physical until now and obeying the "Cartesian dichotomy" as a particular case among the general "Pythagorean" one, after which physics and mathematics are to be merged or "entangled" (from the viewpoint of physics). However, the CMI formulation of the "Yang-Mills existence and mass gap problem" does not mean those distinctions, therefore being ambiguous.

One can summarize that the enumerated four fundamental mathematical problems, though belonging according to its formulations and implicit subject to absolutely different domains of mathematical cognition, can be anyway unified by its relation to a single meta-mathematical or mathematical axiom sufficient to distinguish unambiguously Hilbert mathematics from Gödel mathematics. The latter is all mathematics until now by default (at least in Modernity granting the Cartesian opposition of mathematics and physics).

Four aspects of the ontological conflict of Hilbert mathematics versus Gödel mathematics can be illustrated by the four enumerated fundamental mathematical problems as follows. Fermat's last theorem is an unresolvable Gödel statement in Gödel mathematics since it shares the same structure as Yablo's paradox (in detail in *Penchev 2021 March 9*). Consequently, Wiles's proof cannot but involve an inverse Grothendieck universe in the rigorous meaning above. Anyway, Fermat's last theorem is provable in Hilbert mathematics (Penchev 2022 June 30; 2022 May 11; 2021 March 9), which can furthermore hint at the way to be proved in "Fermat arithmetic", which follows a modified Husserlian "epoché to infinity", in which unresolvable Gödel statements do not exist at all. Indeed, all powerful contemporary mathematics relying on set theory might not be involved, but this is not crucial for the proof.

On the contrary, the four-color theorem and Poincaré's conjecture being topological statements (thus not needing arithmetic directly) seem to be valid only in Gödel mathematics rather than in Hilbert mathematics[13]. However, both rely on a fundamental property shared by the relation of Gödel mathematics to Hilbert mathematics, which can be called the "homeomorphism of a bit of information".

Finally, the "Yang-Mills existence and mass gap problem", being a physical problem able to be generalized as mathematical if one considers its origin, can be directly linked to the relation of Hilbert and Gödel mathematics or that of Gödelian and non-Gödelian mathematics in the framework of the former. Even more, its eventual solution depends on the context in which it is formulated: whether physical or mathematical (meant in the standard sense of Gödel mathematics by default until now).

So, one can utilize the metaphor of a new "dimension" introduced by Hilbert mathematics to Gödel mathematics, and then one can demonstrate (for example, by means of the enumerated four fundamental mathematical problems) that the addition of the new "dimension" at issue supplies one more "degree of freedom", which allows for their eventual solutions to be exceptionally simplified. The metaphor of that new dimension can be furthermore interpreted philosophically or

[13] The proof of the four-color theorem by enumerating a finite set of all possible cases is valid also in Hilbert mathematics and this observation will be discussed in detail further.

ontologically, in a Pythagorean manner so that the physical world is granted to be a particle case of mathematics if it is Hilbert mathematics, and physics and mathematics can be absolutely unified by adding of the new dimension in question.

However, the metaphor of a new dimension of Hilbert mathematics to Gödel mathematics can be understood also literally, i.e., as a rigorous meaning, after which the new dimension is defined by doubling any mathematical structure meant in advance and following the pattern of quantum mechanics forced by its main problem (namely, to describe uniformly both continuous and discrete, quantum changes or quantum-mechanical movements) to use the separable complex Hilbert space being inherently dual. That exact meaning of the new dimension can be also interpreted as "informational", and the theory of information (though historically appeared as an applied and rather technical discipline, but now generalized to quantum information) to be able to unify physics and mathematics therefore underlying both.

II THE NEW INFORMATIONAL DIMENSION OF MATHEMATICS AND THE RESOLUTION OF THE MOST FUNDAMENTAL MATHEMATICAL PROBLEMS

So, one can consider the cited four fundamental mathematical problems as four "case studies", each of them able to demonstrate in a uniform way the fundamental limitation of Gödel mathematics rooted in the general organization of cognition of modernity, after which mathematics and physics are gapped on the two opposite "shores" of the Cartesian abyss of "mind" versus "body", allowing for humankind, supplied by the unique capability of free will, to be the supreme and unique arbiter about the adequacy or not of mental images (or mathematical model in particular) and reality (including physical, being the proper subject of physics). In fact, the concept of "God" (for example in Christianity) can be interpreted as a postulate establishing humankind as "God's vicegerent on earth" to dominate all over the world and over anything in it.

However, this is not more than a convention, prejudice, at that harmful, since it prevents the further development of humankind's cognition tabooing huge and crucially prevailing domains of possible research for example such as that of "*dark* energy" and "*dark* matter" or the way for the human mental free will to be transformed into physical action as a fundamentally inaccessible *"black* box", without any other reason for the taboo at issue originating from that prejudice. If humankind would like to study those tabooed areas (and many others obeying it) that irrational taboo or prejudice should be removed from them. As to the pair of physics and mathematics being the proper subject of the present paper, this means to introduce the new informational dimension, in which they are the same.

So, the new dimension of mathematics furthermore being definitive for Hilbert mathematics can be interpreted more or less literally as that of physics and the material world, after which the newly introduced degree of freedom allows for all most fundamental problems of the standard mathematics being Gödel one to be manipulated so easily that their solutions are elementary often even trivial, and the circumstance in question can be demonstrated by the enumerated four ones chosen to represent different aspects of the newly acquired freedom by still one degree of it.

Regardless of that, the new approach though extremely fruitful for mathematics itself needs its emancipation and liberation from the Cartesian organization of cognition in Modernity, in which it is subordinate to reality, in turn being opposed to mathematics in default, and thus to the material world and physics, which it supplies with more or less relevant tools, so-called mathematical models, the applicability of which is absolutely necessary to be checked by experiments granted to be always local, causal and thoroughly repeatable, therefore crucially restricting the "free flight of the imagination and the pure fantasies" of mathematics by the "chains of materiality".

Quantum mechanics, being a physical and thus experimental science, is forced to release its mathematical models from their repeatability replaced by probability (density or not) distributions, their causality by probabilistic casualties, and their locality by inherent nonlocalities. Nonetheless, that "excessive freedom" claimed by quantum mechanics met the desperate resistance of so great physicist and minds as Einstein and his sardonic mocking by pejorative epithets as his picturesque expressions such as "spooky action at a distance", "God playing dice", "physics of ghosts", the statement that better for him not be a physicist if quantum mechanics is the case, etc.

Anyway, Wolfgang Pauli managed to offer a Solomonic decision, his "particle paradigm" of unitarity and energy conservation, which "classical quantum mechanics" including the Standard model accepts and obeys. Its compromise consists in the "limited sovereignty" of quantum mechanics obeying the universal law of uniform time and energy conservation implying in its own territory unitarity and all physical quantities to be Hermitian operators. However, the 2022 Nobel prize in physics establishes for that "spooky action at a distance" to be the phenomenon entanglement very well confirmed experimentally, and factually legitimates that "God really plays dice" where those dice turn out to be qubits, units of quantum information, the theory of which is forthcoming to replace classical quantum mechanics, Pauli's paradigm, or the Standard model, all relying on energy conservation: due to quantum information conservation generalizing the former.

The metaphor of Lord Kelvin's "clouds" (originally at the horizon of physics in the eve of the 20th century) can be revived to contemporary physics, but in relation to dark matter, dark energy, entanglement, quantum information, and quantum gravity. All of them are inconsistent to classical quantum mechanics, Pauli's "particle paradigm", the Standard model, energy conservation, unitarity and "Hermitianity". So, one can suggest quite reasonably that the theory of quantum information after relevantly generalizing classical mechanics will manage to resolve all of them even in a uniform way. However, those considerations refer to physics properly while one should mean rather the new physical dimension of mathematics, therefore turning to be Hilbert mathematics due to the same, in the context of the present paper.

Meaning them from the proper, formal and abstract viewpoint of mathematics, the new dimension consists only in doubling: so, it seems to be rather informational. Indeed, any bit of information admits to be interpreted as a formal and abstract doubling of whatever including any mathematical structure or mathematics as a whole, and then, the choice of either copy starting from the initial state of a single and unique copy of the same; furthermore:

The relations between propositional logic, set theory, and arithmetic, being crucial for the foundations of mathematics[14] (for example, as both Gödel completeness and incompleteness

[14] There exist many papers, the subject of which are the foundations of mathematics and their relations to other problems. Some of them closer relatable to the approach of the present paper are for example:

papers demonstrate convincingly), can be also represented though formally and abstractly as a bit of information, after which arithmetic can be considered as either "half" of logic, in turn identified algebraically with set theory as the same Boolean structure (in detail in *Penchev 2023 January 3*). Anyway, propositional logic and set theory are differentiated meaningfully from each other as follows: propositional logic is granted to be zero-order of mathematics by default, and set theory represents the class of all first-order logics, or in other words, the class of equivalence of all possible mathematical theories.

Once the structure of a bit of information has been postulated for the relations of arithmetic, propositional logic, and set theory, a few more conclusions follow immediately. Actual infinity after Cantor is to be identified just as a corresponding doubling of finiteness (for example granted in Peano arithmetic for the axiom of induction in advance), or respectively as the new dimension due to the postulated gap between the two copies of finiteness, definitively necessary for infinity.

Furthermore, if one accomplishes the aforementioned "epoché to infinity", now in relation to the doubled copy of finiteness, he or she is able to restore "Eden before the original sin to have consumed": in fact, the approach of ontology invented already by Aristotle to Plato's philosophy "consumed the original sin" by means of the fundamental doubling of all things by their ideas. In other words and returning to the interrelations of propositional logic, arithmetic, and set theory, relevant to the foundations of mathematics, the inherent intensionality of propositional logic is able to prevent all paradoxes of Cantor's actual infinity being extensional: an observation inspired Russell (also in a team with Whitehead) for "logicism" or for "Principia mathematica", in fact, only repeating Aristotle's step more than two millennia ago, though now in relation to the foundations of mathematics rather than to philosophy as it was originally.

Then, the new dimension of Hilbert mathematics to Gödel mathematics, above qualified to be "informational", can be not worse interpreted as that of the foundations of mathematics, after which it can be inherently complete overcoming in particular the Gödel dichotomy about the relation of arithmetic to set theory: either incompleteness or contradiction. Indeed, this, after being interpreted by the trivial structure of a bit of information, seems to be obvious just as either alternative of bit is either incomplete to the initial state of both alternatives in a coherent state (for example, that of "Schrödinger's cat") or inconsistent to the other alternative since it can be always seen to be the logical negation of the former.

Still two instructive interpretations (for introducing quantum information in the foundations of mathematics and about the formal structure of Husserl's phenomenology[15]) deserve to be mentioned as relevant to the present context of the "dimension of Hilbert mathematics". Quantum information though historically introduced by quantum mechanics and thus by the separable complex Hilbert space can be proved to be equivalent to that generalization of information which

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Waaldijk 2005; Shapiro (2004); Tazzioli (2003); Holmes (2001); Lavrov 2001; Chaitin (2000); Mancosu (2001; 1999); Marion (1999); Haddock (1997); Majer (1997); Marquis (2013; 1995); Hintikka (1992; 1992a); Couture, Lambek (1991); Cameron (1982); Stigt (1979); Stenius (1978); Kuyk (1977); Kitcher (1975); Hatcher (1972); Zulauf (1969).

[15] The relation of Husserl's philosophy to philosophy of mathematics or to Hilbert's viewpoint are considered in a series of papers (e.g., da Silva 2016; 2000; Haddock 1997; Majer 1997; Mahnke 1977).

is relevant to infinite sets and series (Penchev 2020 July 10). Meaning also the interpretation of infinity by two copies of finiteness and the gap between them, repeating the formal and abstract structure of a single bit, quantum information can be reinterpreted as follows:

A qubit of quantum information corresponds to the two alternatives of a choice and the gap between them as a whole, speaking loosely or in other words, to their coherent state before the choice of either of them. That rather extraordinary, but quite consistent understanding of a qubit and thus that of quantum information (since its units are qubits) can be visualized by "Schrödinger's cat". Of course, both "dead state" and "alive state" of it, regardless of either of which is unambiguously determined after the door of its box has been opened, constitute a bit of information.

Nonetheless, Schrödinger's cat itself or by itself (i.e., before the door to have been opened) represents a qubit of quantum information, or similar to any other quantum entity, for the allegory of which it can serve. Even more, the "state of the door of the box", either open or not, is still one bit of information, but complementary to the former bit after verifying the state of the cat "either dead or alive" since it makes sense only after the door has been opened. On the contrary, if the door is yet closed, it does not make sense, and the relevant information is quantum: a qubit corresponding to the coherent state of the dead-and-alive cat therefore being both dead and alive "simultaneously" (the quotation marks are necessary because that "simultaneously" refers to the state "before time itself" to appear as to the cat though it runs normally "out of the box").

One can immediately observe an absolute symmetry between a bit consisting of two complementary elementary oppositions (namely "before the choice - after the choice", on the one hand, and "the one alternative after the choice - the other alternative after the choice, on the other hand), and a qubit consisting of two bits visualized by "Schrödinger's cat" as the bit relevant to the door of the box, on the one hand, and that referring the state of the cat itself, either "dead" or "alive" after the door has been opened, on the other hand.

Thus, if the two elementary oppositions of a bit are transformed into two bits, that initial bit is transformed into a qubit; also vice versa: if the two complementary bits of a qubit are reduced to two bits, the initial qubit is in turn reduced to a bit. Speaking loosely, one might say that a qubit is a meta-bit, or respectively, that quantum information is meta-information. Then, duality inherent to Hilbert space[16] (i.e., whether the separable complex Hilbert space of quantum mechanics or the qubit Hilbert space of quantum information) is necessary and essential for representing the complementarity of a level and its corresponding meta-level in an idempotent way.

Another corollary about the mediated relation of a qubit to any elementary opposition (relevant to it) follows now. If a bit means an opposition and a meta-opposition together though complementary to each other, and a qubit means a bit and a meta-bit analogically together and

[16] The duality of Hilbert space embodies the much wider conception of duality in mathematics at all, in physics a and then in philosophy of mathematics and physics, and even in ontology (e.g., Wu 2018; Bokulich 2017; Giuffrè, Idone, Maugeri 2005; Luis 2004; Emerson 2003; Bandyopadhyay 2000; Maurin 1988; Allen 1977; Massey 1977); also in relation in its link to complementarity in quantum mechanics (e.g. Bokulich 2017; Corfield 2006; Giuffrè, Idone, Maugeri 2005; Luis 2004; Bandyopadhyay 2000; Kuyk 1977).

complementary to each other, than a qubit can be interpreted as a meta-meta-opposition, and consequently two options are possible to elucidate that “meta-meta-relation”: for example, the one is according to propositional logic after the idempotency of negation where a qubit should be identified as an elementary opposition, and the other and quite different one is according to arithmetic where a qubit is at two levels higher than any opposition relevant to it. Whatever is the case, quantum information is related to the foundations of mathematics relevantly to its “three whales”: set theory, propositional logic, and arithmetic.

One can utilize the following illustration by the Schrödinger equation of “Schrödinger’s cat”. The time derivative of the wave function of “Schrödinger’s cat” (in its “left side”: of course conventionally, following the usual way of its expression) is equated to the “space derivative” (i.e. Laplace, or nabla squared operator) of the wave function of “Schrödinger’s cat” (in its right side, again conventionally, and granting for the corresponding “potential field” to be zero). Then, the sense of that equation can be interpreted to be the trivial identity of the same opposition “dead - alive” though notated differently in either side: as the time derivative of the state after the door has been open versus the space derivative of “Schrödinger’s cat by itself” i.e. before the door to have been opened.

One can assure that the above complicated allegory by “Schrödinger’s cat” corresponds to the standard sense of the Schrödinger equation, after which its “right side” means the results of the “apparatus” (to which time derivative and time itself make sense since the apparatus obeys classical mechanics) equated to the state of the measured or investigated quantum entity “by itself” (i.e., corresponding to the state of “Schrödinger’s cat” before the door of the box to have been opened), meant in the “right side” of the Schrödinger equation; indeed trivially: “either dead or alive” (after the door is opened) = “either dead or alive” (the door is yet closed).

The other instructive interpretation refers to the formal and abstract meaning of Husserl’s phenomenology after the theory of information or quantum information. One can speak of a “philosophical bit” of information sharing the same formal and abstract structure. The aforementioned “epoché to infinity” illustrates it very well: if one abandons the distinction between reality and its mental image therefore possessing the structure at issue, that of a bit of information, a distinction invented yet by Plato more than two millennia ago, whether after Husserl’s “phenomenon” or Hegel’s “synthesis”, can notate equally well the return into the initial state before the choice relevant to any bit of information. Both of them realized that approach as logical, accordingly, in the framework of “Logical investigations” and then generalized philosophically to Husserl’s phenomenology, on the one hand, or in “dialectical logic”, on the other hand (i.e., after Hegel rather than Husserl).

The intervention of logic in both cases is not occasional. It repeats the ancient discovery of Aristotle that the inherent intensionality is able to overcome Plato’s dichotomy, doubling, and extensionality. In fact, Russell’s logicism also repeats the same solution to the particular case of the foundations of mathematics. However, the formal and abstract structure of information is absolutely sufficient for embedding that philosophical idea, for example in relation to the distinction of mathematics and physics returning both into their initial state of their unity before the "initial sin" of the choice of either of them to have been "consumed", in fact, revealing the vast horizon of investigation of the "dark phase of the universe" inseparable of its "light phase" identified with the subject of physics until now.

III WHY MANY OF THE MOST DIFFICULT PROBLEMS OF CONTEMPORARY MATHEMATICS ARE UNSOLVABLE IN GÖDEL MATHEMATICS

Meaning the so sketched context of Hilbert mathematics, in which Gödel mathematics can be identified as Gödelian mathematics, one may consider the reason for the most difficult problems of contemporary mathematics (exemplified by the four ones in the present paper) not to be resolved and originating from the existence of the immense domain of all Gödel insoluble statements once the three "whales" of arithmetic, set theory and propositional logic are granted in advance and thus the Gödel dichotomy about the relation of arithmetic to set theory is unavoidable. That "insoluble" area can be defined rigorously as the complement of arithmetic to set theory if they are granted to be consistent to each other as propositional logic needs.

If one means that the axiom of induction (e.g., in Peano arithmetic) is either the logical negation of the axiom of infinity (e.g., in ZFC set theory) or it defines the special concept of "infinite set" in a way to be insurmountably gapped from any finite set, the abyss between them is sufficient to place and insert all insoluble statements whether already formulated or yet not. In other words, the necessary condition for all of them is both arithmetic and set theory to be relevant to them though in one or another sense. This observation can be tried on the enumerated four fundamental problems:

In fact, Fermat's last theorem does not need set theory, respectively the concept of actual infinity, to be formulated. So, it by itself is not necessarily relevant to any Gödel insolubility Anyway, it is involved factually and historically due to the following reasons. It might not be resolved arithmetically more than two centuries before Cantor's set theory appeared. Of course, that fact does not imply that it cannot be resolved only arithmetically at all, but that nobody had guessed how to do it for more than two centuries.

Then, all weapons of contemporary mathematics, thus originating from both arithmetic and set theory, were utilized for proving Fermat's last theorem: and finally, Andrew Wiles (1995) managed to infer it as a corollary from the modularity theorem situated just on the bridge between arithmetic and set theory; or more precisely, the discrete modular forms on the shore of the former and the continuous elliptic curves for the latter. So, though Fermat's last theorem by itself does not involve any Gödel insolubility, Wiles's proof, furthermore being the only one established and confirmed until now, does this.

Even more, one can easily demonstrate by means of Yablo's paradox (Penchev 2022 May 11) that Fermat's last theorem is a Gödel insoluble statement in the framework of Gödel mathematics. Consequently, Wiles's proof cannot but go out of it, for example into an inverse Grothendieck universe as this is sketched above. The problem of how Fermat's last theorem might be proved only arithmetically, therefore excluding any Gödel insolubility, unavoidable once both arithmetic and set theory have been used for its proof, remains, though.

Then, Hilbert arithmetic underlying Hilbert mathematics, thus going out of Gödel mathematics can be involved as a "Wittgenstein ladder" (thus intended to be removed in the ultimate construction) for the objective of an eventual, purely arithmetic proof of Fermat's last theorem. The idea consists in its proof in Hilbert arithmetic (just as Wiles's proof involving an inverse Grothendieck universe, but explicitly unlike it), which can be reduced to an image of it only within "Fermat arithmetic" after a Husserlian "epoché to infinity" by means of the nonstandard bijection of Hilbert arithmetic into Peano arithmetic. The proof of FLT in Fermat arithmetic is rather elementary once the nonstandard bijection has been granted in advance (e.g., Penchev 2022 May 11) as admissible to be used.

Though Hilbert arithmetic crucially facilitating the proof can be ultimately removed as a "Wittgenstein ladder", one can be inspired not to use Hilbert arithmetic at all (that is: even not as "Wittgenstein ladder") trying to restore hypothetically the "lost proof of Fermat" claimed by himself (for example by the joint arithmetic and geometric concept of "natural volumes" accessible to Fermat's age and defined as follows: $V(x) = x^n$ where "x" are natural numbers, and "n" are volume dimensions therefore also natural numbers[17]).

One can conclude that Hilbert mathematics to the problem about FLT is only a heuristic tool, a "Wittgenstein ladder" which can be even absolutely missing, including even as an only auxiliary instrument if one manages to restore the "lost proof of Fermat". This is a possible approach for utilizing Hilbert mathematics to the most difficult problems of Gödel mathematics and exemplified by the first of the enumerated four puzzles: Fermat's last theorem. The next one, Poincaré's conjecture, also resolved positively, may illustrate another way for Hilbert mathematics to assist Gödel mathematics.

Poincaré's conjecture is a topological statement, thus immediately needing only set theory for its formulation rather than arithmetic and in a way opposite to Fermat's last theorem. Perelman's proof, a mirror image of Wiles's one, involves, anyway, arithmetic since it uses the concept of information. One can question whether information needs arithmetic necessarily: rather "yes" since the unit of information, a bit, is discrete unlike, for example, the unit of distance relevant to geometry.

However, one can object that, following Kolmogorov's definition of information (rather than Shannon neither "Kolmogorov complexity"), it can be defined as the relative entropy of a probability density distribution to another: thus, implicitly involving quantum information and set theory in the final analysis. In fact, information is a "Centaur-like" concept able to unify the discrete and continuous, or arithmetic and set theory as this is elucidated in detail above. Perelman's proof needs this property of information and just it implies the relevance of the proof. So, one can suggest that he involved, though indirectly, a Grothendieck universe and thus Hilbert arithmetic just as Wiles did for Fermat's last theorem only by virtue of relying on "information".

As this is demonstrated above, Poincaré's conjecture can be linked naturally to special relativity (though the former is topological, and the latter is metrical) by the topological properties of Minkowski space, since special relativity postulates for it to be equivalent to Euclidean space.

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[17] In more detail in: *Penchev 2021 March 9*.

However, one can notice immediately that special relativity utilizes the one "half" of it: its "imaginary area". Topologically, it is different from Euclidean space though both are opened because Euclidean space is tridimensional and "convex", and Minkowski space is four-dimensional and "concave", thus its imaginary area including.

If one utilizes the topological representation of the tridimensional unfolding of the imaginary area in question, being physically interpretable as the propagation of a light wave in space during time (in the dimension of which the "unfolding" is done), the "agglutination" of the unfolding in the ends of "plus infinity" and "minus infinity" differs from the unfolding of the four-dimensional imaginary area of Minkowski space for Euclidean space.

One may relate that proper geometric representation to both arithmetic and set-theoretical ones in order to be relevant to the Gödel dichotomy about the relation between them: either incompleteness or contradiction. One can use the axiom of choice to enumerate all elements of the unfolding by all natural numbers. Then, the Gödel statements are to be situated "in infinity", which is to be interpreted as one more, and also infinite area rather than as a "point" or "cut", in which the unfolding is to be "glued". So, one can observe the way in which Poincaré's conjecture also touches the fundamental incompleteness of Gödel mathematics: one is to add the real area of Minkowski space, physically interpretable as the propagation of the same light wave propagation backwards in time and thus able to "glue" the unfolding of the other imaginary domain: in order to be homeomorphic to Euclidean space.

Hilbert mathematics by means of Hilbert arithmetic does not suffer from the same problem or from the Gödel incompleteness generally. Indeed, it adds a dual (and anti-isometric) "twin" of Peano arithmetic, by which it is able to enumerate also the entire real domain, or respectively, all possible different Gödel insoluble statements in the common case. Nonetheless, one can realize that dual counterpart as referring to the physical world by itself so that the one domain of Minkowski space (whether "real" or "imaginary") to be associated with the mathematical world, and the other one with the physical world: both opposed to each other in Gödel mathematics, but unified in Hilbert mathematics.

Then, one may understand why "information" rather than any other concept is necessary for Perelman's proof of Poincaré's conjecture: it is able to be both physical and mathematical simultaneously. If one manages (as in fact Grigori Perelman did) to equate the "one kind" of information relevant to the mathematical world to its "other kind" for the physical world, both ends of the unfolding of Euclidean space into Minkowski space will be rigorously and reliably "glued" to each other, and Poincaré's conjecture proved (what Perelman did). The proof involves Hilbert mathematics implicitly and indirectly, by means of "information" able to be both physical and mathematical and thus to glue the unfolding of Minkowski space in order to restore Euclidean space in a homeomorphic way.

Observing the "forest" of how Poincaré's conjecture is proved rather than its "separate trees" in detail, the following essence can be extracted: the homeomorphism of the state of any bit "before choice" (exemplified by Euclidean space) in relation to its state "after choice" (for the two domains of Minkowski space). In fact, though Hilbert mathematics is involved, the same statement is valid only to its Gödelian particular case, but invalid in the common case. Indeed, if the two alternatives of a bit overlap each other in any finite area or any finite area appears between them (respectively the same hypothesis repeated to the real and imaginary regions of Minkowski space), Euclidean space would not be homeomorphic to the unit 3-sphere just because of that finite area which will miss in it, but will be available in the unfolding by Minkowski space, in which the light cone would be an additional intermediate, third region in comparison with the "flat" case of Minkowski space in a usual sense.

That penetration into the essence of Poincaré's conjecture as the fundamental homomorphism of a bit of information is a natural bridge to the four-color theorem in order to be unified. Indeed, it can be proved very easily or even trivially in Gödelian mathematics, or respectively in Gödel mathematics, not touching the problem about its validity in Hilbert mathematics in general:

Any map, whatever it be, is situated on the plane where Cartesian coordinates starting at any point of the plane are always possible. Then one considers the number of colors separately in both abscissa and ordinate and any point: they are obviously always two and if they are different for either the abscissa or the ordinate, they are totally four. Meaning that observation, one can use *reductio ad absurdum* to prove elementarily the theorem. One admits that the map needs any fifth color at a certain point: this implies a third color to appear in the abscissa or the ordinate to distinguish two neighbor areas in either of them. Since the latter is impossible, the former suggestion for the necessity of any fifth color is also false.

Consequently, one means a topological interpretation of the Cartesian axes (for example for two neighbor areas to be defined in either of them), and set theory and thus actual infinity are involved though implicitly. Nonetheless, one can enumerate the number of two dimensional "countries" on the map therefore utilizing the axiom of induction (e.g., in the framework of Peano arithmetic) and then establish that they are always a natural number and thus finite. A Gödel incompleteness statement will be able to be articulated thus questioning the necessity of some fifth color in general since the number of countries on the map is a natural number, but the set of all points of the plain is infinite: not even countable. So, at least a point obeying the Gödel insolubility will belong and will not belong to some "country" at the same time, or on the contrary, will belong to two countries simultaneously. So, that point will need some fifth color on the map, respectively some third color, whether on the abscissa or on the ordinate, to be distinguished. So, the above elementary proof will not be valid in Gödel mathematics immediately.

However, if one assists Gödel mathematics by Gödelian mathematics, the problem appearing between the always finite number of countries of the map, on the one hand, and the actual infinite set of all point on the plain, on the other hand, can be immediately resolved by involving a second and dual plane and respectively, another pair of Cartesian coordinates also dual to the former abscissa and ordinate. Then, any point of the plane (in the framework of Gödelian mathematics) will belong either to some country of the one map on the one plane or to its dual counterpart: i.e., to its dual "country" of the dual map on the dual plane.

Even more, following the pattern for Fermat arithmetic by "epoché to infinity" one can create a "Cartesian map" obeying the same "epoché to infinity" so that the above proof of the four-color theorem is directly valid to it since the two dual "Gödelian maps" (i.e., the sense above) are naively identified to be the same on the "Cartesian map", just by virtue of which one could not yet distinguish the natural number of countries on the map from the infinite set of points on the plain.

Furthermore, the two dual coordinate systems constitute a qubit of quantum information, and both abscissa and ordinate as two possible alternatives of either dual (but single) coordinate system are a bit of information, and then repeat or interpret the "Schrödinger equation of Schrödinger's cat" (above) as a justification of the solution on the "Cartesian map" by that on the "Gödelian map" (i.e., consisting of two dual maps) since it equates them. The so elaborated proof of the four-color theorem would not be valid in Hilbert mathematics in general where a fifth color seems to be necessary to paint "no man's lands" or respectively "condominium lands (like Andorra)". In fact, this suggestion is false, but not by virtue of the above consideration relevant only to the "flat case" of Gödelian mathematics in the general framework of Hilbert mathematics.

However, the machine proof, which relies on the complete enumeration of all possible cases for "countries" to be neighbors on the map, is valid in general in Hilbert mathematics and here is why. It reduces the set of all possible maps to a *certain finite* set of all possible cases (at that enumerated expressly) for countries to be neighbors on the map thus needing different colors to be distinguished for each other by their different color. So, the problem about the relation of the always finite, but uncertain finite number of countries on the map and the actually infinite, even uncountable set of all points of the map, thus unavoidably obeying the Gödel dichotomy, can be prevented after reducing the issue. Indeed, no incompleteness can appear between an exactly determined finite set being absolutely certain by enumerating all elements of it one by one, what the set of all possible cases of neighborhood is, on the one hand, and the actually infinite set of all map points, on the other hand.

So, the exceptionally bulky and clumsy proof consisting of the software programs to prove each case of any possible neighborhood by means of computers, which no human is able to trace it thoroughly from the beginning to the end, is anyway valid in Hilbert mathematics rather than only in Gödelian or Gödel mathematics as the proof sketched above by both abscissa and ordinate "cuts", to which the Gödel dichotomy is quite relevant. The latter proof, though very simple and elegant, is not general. On the contrary, the former proof in the common case is unverifiable step by step by any human intellect. The case study of the four-color theorem to the generalization of Gödel mathematics as Hilbert mathematics demonstrates only the eventual option of a "human" proof of it, but at the cost to be relevant only in a particular case. Anyway, one can hypothetically admit that will be ever able to suggest a relevant "human" proof also of the common case.

The Yang-Mills existence and mass gap problem being unresolved yet and so fundamental to be one of the seven CMI Millennium problems is able to visualize the importance of the different context of Hilbert mathematics versus Gödel (respectively Gödelian) mathematics, in which it can be interpreted. Another peculiarity of it is that it is the mathematical generalization of a physical hypothesis relevant to the Standard model as well, therefore embodying the way in which Hilbert mathematics considers the physical world to be its particular case. Indeed, if the latter is the case (as it claims), just mathematical hypotheses thus relating to the common case of Hilbert mathematics are to originate from its particular case of physics.

However, the Yang-Mills existence and mass gap problem can be resolvable positively if it is interpreted in the physical case of Hilbert mathematics, but negatively in Gödel (respectively Gödelian) mathematics: at that, consistently in both cases. If the latter "half" of the problem, that of the mass gap, corresponds unambiguously to the eventual positive solution in the non-Gödelian Hilbert mathematics and to the eventual negative solution in the Gödelian Hilbert mathematics, its former "half", that the Yang-Mills existence is more sophisticated (as this is elucidated above).

Nonetheless, the contribution for the solution in Hilbert mathematics (whether negative or positive) is doubtless. Even more, it assists Hilbert arithmetic to make clear for itself why the qubit Hilbert space as the dual counterpart of Hilbert arithmetic in a narrow sense is to appear: just by virtue of the Yang-Mills existence rather than due to postulating. On the contrary, whether the negative solution or the positive one is predetermined in advance by the value of the "distance between finiteness and infinity"; accordingly, zero or nonzero.

IV THE GENERAL STRUCTURE OF THE INSOLVABILITY IN GÖDEL MATHEMATICS AND THE CONTRIBUTION OF THE NEW DIMENSION INVOLVED BY HILBERT MATHEMATICS

Meaning those four case studies, one can suggest that the Gödel incompleteness (since the alternative of contradiction is excluded in definition after granting propositional logic in advance) is the same reason shared in the enumerated four problems and then conjectured to be relevant to the most difficult problems of contemporary mathematics. So, the common insolvability model in Gödel mathematics is to refer to the incompleteness at issue because the essential mathematical problems need both arithmetic and set theory whether still in their formulation or at least for its solution.

Furthermore, the insolvability at issue is to be transformable into solvability after passing into Hilbert mathematics as an additional necessary condition for the abstract formula of it. So, one admits the "nonstandard bijection" (suggested in previous papers, e.g., Penchev 2021 March 9), namely "$\{P^{+} \otimes P^{-} \leftrightarrow P^{0}\} \leftrightarrow P$" where P means Peano arithmetic, P^{+}, P^{-} are the two dual anti-isometric copies of Peano arithmetic in Hilbert arithmetic, so that the pair of them corresponds to set theory. Then, all statements of Gödel incompleteness are to be related to the dual copy of Peano arithmetic, complementing the initial copy of it to set theory. Accordingly, the nonstandard bijection itself demonstrates how that kind of insolvability is transformed into solvability only by virtue of passing into Gödelian mathematics (i.e., in the framework of Hilbert mathematics).

One can further test how or how far that a formal and abstract model fits to the enumerated four problems. Fermat's last theorem is an unsolvable problem in Gödel mathematics since all cases belonging to the dual counterpart of Peano arithmetic cannot be resolved whether positive or negative. So, Fermat's last theorem is an unsolvable problem in Gödel mathematics and thus Wiles's proof goes out of it necessarily, which may be shown by means of Yablo's paradox (in detail in *Section VI*). On the contrary, the nonstandard bijection implying a relevant solution in Hilbert mathematics implies also the necessary existence of its solution in "Fermat arithmetic" defined above (as well as in other papers) by “epoché to infinity”, after which the question whether the number of all natural numbers is finite or infinite is abandoned.

Fermat’s last theorem will not be solvable in the common case of Hilbert mathematics if one thinks as follows. The suspended “excluded middle” of intuitionistic mathematics as to the pair of finiteness and infinity suggests “no man’s land” between them, for which FLT seems to be uncertain. However, nobody knows whether the approach for the reduction of all cases to a certain finite set, the elements of which can be enumerated one by one and demonstrated by the “machine” proof of the four-color theorem might not be repeated also to FLT to supply its complete proof in the common case of Hilbert mathematics. Thus, one can generalize that observation into the conjecture that any problem unsolvable in Gödel mathematics for touching the Gödel incompleteness is anyway solvable in Gödelian mathematics and again unsolvable, but already in Hilbert mathematics.

One can call the explicit reference to a certain finite set (as in the case of the “machine” proof of the four-color theorem) the “argument of finitism” in short. Then, the above conjecture can be expressed so: the argument of finitism is always applicable to both Gödel mathematics and Hilbert mathematics simultaneously. However, many fundamental problems unsolvable in Gödel mathematics turn out to be rather easily resolved in the “flat” case of Gödel mathematics in the framework of Hilbert mathematics just by virtue of the addition of the new “informational dimension” granting it to be zero. Nonetheless, the same problem in the common case of Hilbert mathematics, thus at an arbitrary nonzero value of the informational dimension is unsolvable as in the initial case formulated by Gödel mathematics. In other words, many fundamental mathematical problems are rather solvable under the additional admission for infinity to be a second finiteness literally repeating the initial first one and gaped from it therefore both constituting the new “informational” dimension with the formal and abstract structure of a bit information.

Poincaré’s conjecture seems to exemplify the nonstandard bijection directly rather than only to touch it or to be relatable to it: if the nonstandard bijection is considered topologically, i.e., as a homeomorphism, Poincaré’s conjecture is a corollary from it. The topological interpretation of the nonstandard bijection relies on the axiom of choice, but it supplies only for the nonstandard bijection to follow from its topological interpretation, speaking loosely, as the nonstandard model of its topological interpretation.

A question is whether the reverse statement of the axiom of choice is also valid if it holds or the reverse statement is to be considered as a separate axiom. The answer is not unambiguous. For example, one can grant that the axiom of choice is applied and an element is chosen from an arbitrary set therefore satisfying a certain property being the characteristic one for the set at issue. May the choice be canceled without any sequel? Maybe “Yes”, in the framework of the standard mathematics which is Gödel mathematics: the choice mediates between arithmetic and set theory in both directions equally well. The element at issue can be enumerated after the choice and thus a certain natural number (being always finite according to the axiom of induction) may correspond unambiguously to it, but it only belongs to the set whether infinite or finite before the choice.

However, the choice, which is made, is a certain amount of bits of information. One can distinguish the state before the choice from that after the choice has been canceled by the quantity of information since it cannot be analogically canceled: on the contrary, it will be doubled after the made choice has been canceled after that. The concept of information is essential for Hilbert mathematics being situated in its foundations. Meaning that, one can doubt whether the reverse axiom of choice is also always valid in its framework.

However, the aforementioned eventual inference of Poincaré's conjecture from the nonstandard bijection would rely just on that questionable "reverse axiom of choice". Perelman's approach for its proof can be now reinterpreted as needing the exact doubling of the information due to the made choice after its cancellation. So, it implicitly realizes Gödel mathematics as Gödelian mathematics (i.e., as the "flat" case in Hilbert mathematics, the "flatness" of which consists just in the above "doubling" of information of any choice after its cancellation).

Thus, Perelman's proof should not hold in the common case, but only in the "zero" case of Gödelian (Gödel) mathematics. One can conclude that the brilliant Russian mathematician has literally accomplished (though implicitly) the conjecture above, referring to the option for proving fundamental mathematical problems by their isomorphic reinterpretation from Gödel mathematics into Gödelian mathematics. However, one can rather difficulty imagine the eventual "argument of finitism" to Poincaré's conjecture (unlike it in the framework of the "machine" proof of the four-color theorem). Thus, it remains a problem in Hilbert mathematics unlike the four-color theorem. Furthermore, one can attempt to generalize Perelman's proof stating that Poincaré's conjecture is false in the common case of Hilbert mathematics. On the contrary, if one invents somehow to apply the argument of finitism, it would be true in Hilbert mathematics as well.

The case of the four-color theorem is opposite to that of Poincaré's conjecture as to the relation of Gödel mathematics to Hilbert mathematics just by virtue of the argument of finitism underlying its "machine" proof: the same statement in Gödelian mathematics (i.e. in turn in the framework of Hilbert mathematics) admits only a particular, but "human" proof, furthermore very elementary, also following immediately from the nonstandard bijection as Poincaré's conjecture. It allows for the rigorous distinction of finite proofs referring to all natural numbers, which are finite only by virtue of the axiom of induction, on the one hand, from proper finitist proofs outlining from a certain finite set thus not needing the axiom of induction in order to be finite. That proper finitist proof seems to be always valid in both Gödel mathematics and Hilbert mathematics, unlike the former finite proofs valid in Gödelian mathematics and thus in Gödel mathematics, but not in the general case of Hilbert mathematics (which does not mean that the corresponding theorems are necessarily false in Hilbert mathematics).

The insolvability of the Yang-Mills existence and mass gap problem is very interesting and it is due to a few reasons. The first of them consists in the fact to be yet unresolved, furthermore being one of the seven CMI Millennium problems. So, the considerations in the present paper can serve as heuristic. The other reason is that the Yang-Mills existence and mass gap problem is a mathematical generalization of a few physical theories in the framework of the Standard model. So, its origin suggests one of the main ideas of both Hilbert arithmetic and mathematics, according to which the physical world is a particular case of Hilbert mathematics therefore resurrecting a new form of Pythagoreanism relevant to our age.

The third and maybe most essential reason relates to the structure of Hilbert arithmetic underlying Hilbert mathematics and supplying its completeness versus the Gödel incompleteness. Hilbert arithmetic in a narrow sense is Peano arithmetic doubled by its dual anti-isometric counterpart. So, the former can represent (e.g.) the infinitesimal local aspect versus the finite global aspect both being inherent for continuum or continuity in a "Hamiltonian language". Then, the qubit Hilbert space is only postulated as the counterpart of Hilbert arithmetic in a narrow sense so that any unit of the latter is the class of equivalence (an "empty" qubit) of the former. The deductive and axiomatic method for building mathematics does not need any reason for the choice of any postulate (including even its direct logical negation):

However, the dogma of experimental science in the standard Cartesian organization of cognition needs additional empirical (whether observational or experimental) confirmations for the choice of one or another mathematical model, which can be again represented by Hilbert arithmetic: mathematics supplies it in a narrow sense, and physics chooses a certain value in the corresponding the empty qubit to be the single "real one". So, mathematics can only postulate physics (as it may postulate any other axiomatics as long as it is in the framework of any consistent first-order logic) in the common framework of Hilbert mathematics.

The Yang-Mills existence and mass gap problem suggests for physics to be inferred from mathematics if the latter is Hilbert one since its first "half" (i.e., the Yang-Mills existence) generates all areas of physics deductively. So, the problem possesses an additional meta-dimension to Hilbert mathematics itself for the justified self-foundation. One can trace back what the Yang-Mills existence and mass gap problem contributes further to the nonstandard bijection: "$\{P^+ \otimes P^- \leftrightarrow P^0\} \leftrightarrow P$"; in fact, nothing is complemented since the nonstandard bijection means any qubit (i.e., "$P^+ \otimes P^-$") and its mapping into the "bits" of Peano arithmetic. In other words, there exists a very close logical and deductive link allowing for the problem at issue to be inferred absolutely rigorously by the nonstandard bijection.

V A FEW CASE STUDIES OF UNSOLVABLE PROBLEMS

The meant four case studies are enumerated many times above and each of the next four sections is intended to consider one of them in detail. Of course, the unsolvable problems (or already resolved but very difficult and fundamental) are much more. Which is the reason to be chosen just these four? The present section will try to justify that choice.

Initially, one can introduce the metaphor for "four colors" sufficient for the map of "terra incognita" (for the unresolved problems) to be ever colored although it is an amorphic "white spot" now just any "terra incognita" need be. As the investigation of the four-color problem suggests, the four colors at issue can be unified as the two oppositions of a single bit, and thus as a single bit in the final analysis.

What can make clear the essence of the metaphor is the "teleportation theorem"[18], according to which "two bits" (in fact, two oppositions and thus a single bit) are necessary to be delivered by a classical communication channel obeying not exceeding the speed of light in a vacuum to any teleported "instantly" quantum information in order to be restored by "Bob" the initial quantum state transmitted by "Alice". So, one can conclude that any *local* point in which either Alice or Bob might be (since the existence of both is local in definition and none of them is able to be nonlocal "like God") needs an additional bit to be determined ultimately just as that local point rather than any other.

Then, one can utilize the axiom of choice equivalent to the well-ordering "theorem" so that all space-time to be represented as an unlimited tape of a Turing machine, in each cell of which a bit of information can be recorded or not as just that bit meant by the teleportation theorem. If one considers the complete "tape" of all space-time after set theory, it is an actually infinite set and therefore it can be interpreted as an empty qubit, in which any cell of the Turing machine tape (and thus, a certain value of the qubit of all space-time) can be chosen: for example, as Alice's localization.

Further, one can imagine that qubit transmitted instantly by Alice to Bob by a quantum channel as that "qubit of all space-time" being the same at any point of it including those of both Alice and Bob, whatever Alice's localization would be as the additional bit of classical information, which is necessary to be also transmitted to Bob by a classical channel and thus obeying the restriction of locality by the speed of light in a vacuum, which can be furthermore visualized by the motion from a tape cell to another or by the function successor postulated in Peano arithmetic.

In other words, what is nonlocal and being a qubit can be likened to an infinite set and a certain single element chosen from it; on the contrary, what is local and being a bit is the localization of the chosen bit at issue: it is the same as in the former, "nonlocal" case, but described arithmetically. So, one can suggest that one needs both arithmetic and set-theoretical descriptions of any problem but in a way avoiding the Gödel dichotomy, for example, by means of Hilbert arithmetic and which can be visualized in terms of the teleportation theorem as follows:

Both Alice and Bob transmit to each other a qubit nonlocally and which can be postulated to be the same by virtue of its nonlocality for example as that of all space-time within which both are and both transmit simultaneously the own localization to the other by two different classical channels or by two dual and anti-isometric copies of the same classical channel where both dual and anti-isometric copies of Peano arithmetic belonging to Hilbert arithmetic can be interpreted as those channels.

[18] Teleportation theorem and the phenomena of teleportation are discussed in a series of papers in a sufficiently wide context, thus relevant to the present paper (e.g. Whitaker 2012; Wang, Yan 2011; Hotta 2010; Krauter, Sherson, Polzik 2010; Zak 2009; Furusawa, Takei 2007; Pati, Agrawal 2007; Xia, Song, Song 2007; Liu, Zhang, Guo 2003; Song 2003; Braunstein 2002; Busch, Cassinelli, Vito, Lahti, Levrero 2001; Duwell 2001; Janszky, Gábris, Koniorczyk, Vukics, Adam 2001; Sokolov, Kolobov, Gatti, Lugiato 2001; Stenholm 2001; Laiho, Molotkov, Nazin 2000; Żukowski 2000; Brassard, Braunstein, Cleve 1998; Horodecki, Horodecki, Horodecki 1996).

The fundamental bit of information (Penchev 2021 April 11) can be now interpreted in two dual ways, correspondingly, physically and mathematically: (1.1) the qubit of all space-time versus its local interpretation, in which Alice's and Bob's localizations can be individualized whatever they be; (1.2) Alice's localization versus Bob's localization; (2.1) set theory versus arithmetic; (2.2) the one copy of Peano arithmetic versus its dual and anti-isometric counterpart. Finally, one can notice that the four oppositions (1.1), (1.2), (2.1.), and (2.2) constitute a bit of information and which can described as: (3.1) physics and mathematics unified by Hilbert mathematics versus them opposed and gapped to each other and after which mathematics can be identified as Gödel mathematics; (3.2) the one copy of Gödel mathematics for all standard mathematics until now versus the other and anti-homomorphic (algebraically) copy of Gödel mathematics as the physical world by itself (in a Pythagorean manner).

Then, the four problems are to be so chosen that they are able to be exemplified as the last two oppositions (3.1) and (3.2), namely and correspondingly: the Yang-Mills existence and mass gap problem versus Fermat's last theorem; the four-color theorem versus Poincaré's conjecture. One can assure that the enumerated four fundamental mathematical problems are able to be those fundamental "four colors" able to "paint" the "white spot" of all "terra incognita" of mathematics; indeed: `

The Yang-Mills existence and mass gap problem means really the unification of physics and mathematics in Hilbert mathematics, after which the first "half" (i.e., the Yang-Mills existence) means the doubling of Hilbert arithmetic by its physical counterpart, as what the qubit Hilbert space can be interpreted. The second "half" (i.e., the proper mass gap problem) corresponds to the distinction of physics (to which the mass gap is valid) from Gödel mathematics (to which the mass gap is not valid) both being related to the shared framework of Hilbert mathematics, in which Gödel mathematics is identified with Gödelian mathematics.

The "color" opposite to the Yang-Mills existence and mass gap problem is Fermat's last theorem in the following sense. It is a Gödel insoluble statement in Gödel mathematics, which will be demonstrated in detail in the next section by means of Yablo's paradox. Nonetheless, it is a soluble statement in so-called Fermat arithmetic after "epoché to infinity" being another interpretation of Husserl's original "epoché to reality".

Now, if one realizes Fermat's last theorem ontologically (as Pythagoreanism suggests) and notifying it as "$x^n = y^n + z^n$", the variables "y^n, z^n" are to be divided and opposed as belonging to the "mathematical and physical worlds" distinguishable from each other only for "$n \geq 3$" where qubits correspond to the physical world and the classes of equivalence of all possible values of a qubit, the "empty qubits" are proper arithmetic units. This is a distinction possible only for "$n \geq 3$", after which the equation of them is not more possible since the physical world (to which either "y^n" or "z^n" belongs) is incommensurable to the mathematical world (to which the other counterpart, either "z^n" or "y^n" belongs correspondingly): so their sum "$y^n + z^n$" cannot be ever equated to "x^n" relatable to Fermat arithmetic itself not able yet to differentiate natural numbers from enumerated sets of things. On the contrary, the distinction and thus incommensurability of physics and arithmetic is not relevant yet for "$n = 1,2$" and they can be equated to each other after Fermat arithmetic.

The just sketched philosophical interpretation of Fermat's last theorem allows for explaining the way in which it is to be opposed to the Yang-Mills existence and mass gap problem: the former means the distinction of the physical and mathematical worlds from the viewpoint of mathematics, and the latter considers the same from that of physics; so both are related philosophically to the same.

Once mathematics has been chosen (i.e., after 3.1), the next choice (3.2) is also possible; and it consists, loosely speaking, in the opposition of finiteness in the meaning of the axiom of induction versus finiteness in the meaning of a certain finite set, all elements of which can be indicated directly (unlike the former case where some infinite set corresponds). Those two meanings of finiteness are then correlated with actual infinity by itself as set theory postulates it by the mediation of topology since Poincaré's conjecture and the four-color theorem are both topological problems thus involving actual infinity necessarily.

So, the four "colors" meant by the enumerated four, most fundamental mathematical problems can be schematically reduced to the following: infinity (for the unification of physics and mathematics the Yang-Mills existence and mass gap problem) versus finiteness (for Fermat's last theorem in Fermat arithmetic not yet differentiating physic from mathematics; and if the latter (i.e. finiteness) is the case, the finiteness after the axiom of induction (for Poincaré's conjecture) versus the finiteness of a certain finite set (for the four-color theorem), both meant in the context of topology, i.e. that of actual infinity and thus that of the former member of the former opposition.

VI FERMAT'S LAST THEOREM (FLT)

Wiles's proof of FLT infers it as a corollary from the modularity theorem (known as the Taniyama - Shimura - Weil conjecture before Wiles to have prove it). The essential fact of the modularity theorem in the present context consists in equating discrete arithmetic structures relevant to Diophantine equations (such as modular forms) to continuous geometric structures (such as elliptic curves). So, the approach of Wiles implies the following dichotomy: if it suffers from the Gödel incompleteness (if one grants to be consistent) or it goes out of Gödel mathematics (supposedly, in an inverse Grothendieck universe as above). In other words, there should exist modular forms such that they could not be equated to elliptic curves in the proper and rigorous framework of Gödel mathematics.

One can notice that the modularity theorem faces the same problem as quantum mechanics a long time ago, forced to describe uniformly the discrete quantum entity by itself (for the Planck constant) by the continuous readings of the apparatus obeying the smooth differential equations of classical mechanics. So, quantum mechanics cannot but go out of Gödel mathematics just as the proof of the modularity theorem needs. In fact, it is forced to utilize the same inverse Grothendieck universe being represented as the one of the dual copies of the separable complex Hilbert space involved just for the uniform description at issue.

The idea of Hilbert arithmetic in a wide sense intended to overcome the Gödel dichotomy about the relation of arithmetic to set theory relies also on the qubit Hilbert space, as which the usual separable complex Hilbert space of quantum mechanics can be easily identified. So, Fermat's last theorem can be linked to Hilbert arithmetic following the above associations starting from Wiles's proof of FLT by the modularity theorem and necessity for discreteness and continuity to be uniformly described just as quantum mechanics is forced to do, involving just the separable complex Hilbert space for this purpose. So, one can suggest that Hilbert arithmetic would be a relevant tool for FLT to be attacked (Penchev 2022 June 30; 2022 May 11; 2021 March 9).

Even much more, though Hilbert arithmetic represents in fact an inverse Grothendieck universe thus far transcending Gödel mathematics just as Wiles's proof needs (since FLT is a Gödel insolvable statement Gödel mathematics), it can be anyway used as a "Wittgenstein ladder", that is as a heuristic tool, which can be removed in the ultimate syllogism so that the text of the proof in Hilbert arithmetic to be "translated" in the "language of Fermat arithmetic", in which Gödel insolvable statements cannot exist in principle as far as Fermat arithmetic obeys "epoché to infinity".

The translation from Hilbert arithmetic to Fermat arithmetic relies on the nonstandard bijection and thus on whether it is really a bijection in the final analysis. It contains furthermore a preliminary stage translating from the qubit Hilbert space to Hilbert arithmetic in a narrow sense (which means the "empty" qubit Hilbert space as this is elucidated in detail above). The essence of the FLT proof in Hilbert arithmetic needs only the inspiration for "y^n, z^n" (according to the notations above) to be opposed so that the one (never mind which of both) is considered as belonging to Hilbert arithmetic in a narrow sense and consisting of "units" versus its dual counterpart from the qubit Hilbert space relevant to the physical world and consisting of empty qubits.

Then, the proper translation into the language of Fermat arithmetic needs only the obvious identification of units of Peano arithmetic and empty qubits of the qubit Hilbert space. If one involves the idea of "first-order arithmetic" following the pattern of "first-order logic", and then the class of equivalence of all "first-order arithmetics" as coinciding with Peano arithmetic just as propositional logic as a Boolean algebra is homomorphic to set theory as the class of all possible first-order logics, this is sufficient to justify the translation at issue.

Moreover, one can try Fermat's "lost" original proof involving a generalization of the relation of equivalence, maybe not precise and correct enough to the criteria for rigorousness for a mathematical proof nowadays, but intuitively true and relevant to Fermat's age. It can be notated so: "$n = n\ things$", therefore identifying any natural number with the set of the same number of elements, but considering this in relation to the relation of equivalence rather than to the intuitive definition of "natural number" as usual. Obviously, the "$n = n\ things$" is not a correct relation of equivalence in Gödel mathematics since the natural number "n" is always finite by virtue of the axiom of induction unlike the set consisting of "n" elements, which is actually infinite if it consists of all natural numbers "n".

However, the same interpretation of the relation of equivalency is quite correct in Fermat arithmetic once it obeys "epoché to infinity" implying immediately the kind of equivalence (as in "$n = n\ things$") at issue to be correct and admissible. Involving that understanding of equivalence and complementing it by a slightly modified Fermat descent (both being quite accessible to Fermat himself), one can restore the claimed proof literally in a page or two (Penchev 2020 August 10). The generalization of equivalence as "$n = n\, things$" in fact represents the main idea of Pythagoreanism (that the things are numbers) and also utilized for the proof in Hilbert arithmetic, but in a "naive", "innocent", "untried" way: absolutely suitable for the "Eden of Fermat arithmetic" before the "original sin" of Cantor's set theory to have been "consumed".

One may further notice that the modified "Fermat descent" is rather relevant to the tenet of Yablo's paradox especially if it is applied to FLT to demonstrate that is an insolvable statement in Gödel mathematics (Penchev 2021 March 9), but on the contrary, solvable in Fermat arithmetic. The claim of Yablo's paradox not to be really self-referential is questionable at all, but as if obvious at least at first glance:

The truth of any statement depends of the truth of the next one; or in relation to FLT: the truth of "$x^n = y^n + z^n$" depends on the truth of "$x^{n+1} = y^{n+1} + z^{n+1}$", or respectively the untruth of "$x^{n+1} = y^{n+1} + z^{n+1}$" depends on the untruth of "$x^n = y^n + z^n$" meaning furthermore that the untruth of "$x^{n+1} = y^{n+1} + z^{n+1}$" implies the untruth of "$x^n = y^n + z^n$"

However, Yablo's paradox itself is valid only in Gödel mathematics being a corollary from the existence of Gödel insoluble statements. Indeed, the recursive scheme meant by Yablo's paradox cannot ever reach any actually infinite set, after which Yablo's paradox would be refuted since the recursive scheme refers inherently to natural numbers being always finite according to the axiom of induction. Also starting by any actual infinite set, the modified Fermat descent (emphasizing that even after being modified) cannot reach any finite set for the impossibility to overcome an inverse Grothendieck universe by any recursive scheme by any length.

On the contrary, if one utilizes the Fermat descent (modified or not) in Fermat arithmetic, any analogical unattainability should not appear since it starts from an arbitrary natural number (thus finite) rather than from an actually infinite set as in the former case. In other words, any analogue of a Grothendieck universe cannot appear in Fermat arithmetic since the former needs both arithmetic and set theory and their discernible distinction.

VII POINCARÉ'S CONJECTURE

Poincaré's conjecture has the crucial (in the present context) advantage to be directly interpretable physically, in terms of special relativity, and then from it: to general relativity. The mathematical formalism of the former, Minkowski space (being furthermore homeomorphic to pseudo-Riemannian space relevant to the former) is able to illustrate the physical meaning of Poincaré's conjecture immediately:

Causality is closely "entangled" to the topological equivalence of Euclidean space where are situated all experiments (and even *any possible* empirical experience) and Minkowski space claiming to be equivalent to the former after special relativity and the postulated limit of the speed of light in a vacuum. However, one can notice that the only "half" of Minkowski space, just that one, recognized by special relativity to make physical sense alone, its "imaginary area" is relevant to causality. It is differentiated topologically from the unit 3-sphere and thus, from Euclidean space by virtue of Poincaré's conjecture itself, by the "joining of the two ends" in the case of the 3-sphere to the unfolded case of the imaginary area (alone, i.e. without the dual counterpart of the real domain of Minkowski space).

Then, one can interpret the gap between the two ends after unfolding as homeomorphic to the entire real domain. However, it (interpreted after special relativity) means reverse causality (i.e., where the effect precedes the cause). Nonetheless, this does not generate any contradiction if human experience is restricted to be only within the imaginary light cone and one postulates that what is first during time is the cause, and the second one is the effect:

If both may be first and thus the cause of the other, they are granted to be equivalent. If reverse causality takes place unambiguously (i.e. without its dual counterpart of "normal" causality), it can be anyway represented as the usual "normal" causality after exchanging the cause and the effect in the corresponding empirical description of our experience to that alleged to be "by itself", but unobservable (this mean: out of any possible human experinece).

Then, one can realize a topological reading of causality after both special relativity by Minkowski space and Poincaré's conjecture. Topological continuity can be expressed by the function-successor, defined in Peano arithmetic, and then both mutually rejecting axioms of induction and infinity (respectively, arithmetic versus set theory) are applicable. Reverse causality (and thus, the real domain of Minkowski space) would correspond to the area of discontinuity (the gap between the ends of the imaginary area after unfolding the unit 3-sphere meant by Poincaré's conjecture).

Thus, the physical interpretation of Poincaré's conjecture by special relativity and thus by causality allows for the nonstandard bijection to be rediscovered as the physical principle of causality even after quantum information and the phenomena of entanglement. One can distinguish three kinds of physical phenomena after the reinterpretation of the speed of light in a vacuum from an absolute limit into a boundary between locality and nonlocality (or respectively, between "straight" and "reverse" causality): (1) only local phenomena; (2) only nonlocal phenomena; (3) both local and nonlocal phenomena. Furthermore, all the three kinds of phenomena obey the principle of local observability as a universal and inviolable condition for natural science at all and physics particularly.

Then, all only nonlocal phenomena can be redescribed as ostensibly local under the additional exchange of "cause" and "effect". One can immediately notice that the re-description under the condition for any experience to be local possesses the same formal structure as the nonstandard bijection where only Peano arithmetic is substituted by locality therefore suggesting the option for the identification of mathematical "finiteness" (after the axiom of induction in Peano arithmetic) and physical locality (after the postulate of not exceeding the speed of light in a vacuum after special relativity). Furthermore, the mathematical anti-isometry of the dual copy of Peano arithmetic can be also understood physically as reverse causality.

In other words, Poincaré's conjecture allows for an impressive visualization of one of the main theses of Hilbert mathematics: the unification of mathematics and physics[19]. On the one hand, the

[19] The unification of physics and mathematics is a general idea in contemporary science discussed in various papers (e.g. Cartwright, Giannerini, González 2016; Zade 2016; Palmer 2014; da Costa 2012; Omnes 2005;

nonstandard bijection, once admitted, resolves it rather elementarily, but on the other hand, it can be interpreted physically by means of Minkowski space and then, directly by special relativity. Further, the physical interpretation can be continued back (i.e., returned) to the initial (for the proof) nonstandard bijection demonstrating that it is both mathematical and physical equally well.

Following the same direction of thought, the nonstandard bijection, though being both mathematical and physical, but as a *mathematical* structure anyway, can be supplied by a dual counterpart in turn also being both mathematical and physical, but as a *physical* quantity such as information really able to be both physical and mathematical by virtue of the fact that it is physically dimensionless. For example, information can be formally defined as that quantity which is measured in bits, and a bit is in turn defined as the pair of namesake (or "numbersake" in fact) dual natural numbers or subsets of natural numbers such that the one member belongs to the one copy of Peano arithmetic, and the other one, correspondingly, to the other dual and anti-isometric counterpart.

Then, having the tool of information as well as the interpretation by Minkowski space, one may observe the behavior of information on each of the two sides divided by the barrier of light cone therefore proving that the behavior is the same: that is, information is to be mathematically equal from the two sides of the bound physically interpretable as the limit of speed of light in a vacuum (or as the boundary between locality and nonlocality, mathematically), which implies the transition through the light cone to be topologically continuous. In fact, this is the idea of Perelman's proof if it is represented in terms of the present paper and consistently with its context.

Finally, one can continue back (i.e., again return) Perelman's result to the boundary between the two alternatives of any bit, thus demonstrating that (rather counterintuitively) its two alternatives regardless of the boundary between them are topologically continuous between each other. So, one can utilize the metaphor of the "Berlin wall" between the two alternatives of a bit as a quite artificially built septum within the same "city", the "citizens" of which are absolutely the same regardless of which side of the wall they are. One can go ahead with figuring a homeomorphism of "Berlin before and after erecting the wall": Berlin is the same city, as well as Berlin now: when the wall has been destroyed. That is an illustration of the idea of the homeomorphism of a bit information, underlying Perelman's proof in the final analysis and inferable in turn from the nonstandard bijection as follows:

The nonstandard bijection as any bijection means reversibility necessary for any homeomorphism, but the two directions of that reversibility (unlike the usual one) are complementary or dual to each other: they cannot be ever accomplished simultaneously, but only alternatively: either the one or the other. However, the complementarity or duality of the two directions of reversibility seems to be normal and natural in comparison (for example) with the complementarity or duality of physical quantities or mathematical structures where it looks to be rather counterintuitive and artificial.

Basson 2002; Barrow 2000; Chapline 1999; Wang 1995; Plato 1994; Corsi, Dalla Chiara, Ghirardi, eds. 1993; Requardt 1991; Benioff 1976.

Reversibility unlike any other property or relation is directly decomposable into two opposite irreversibilities intuitively and naturally complementary or dual to each other since it originates from the irreversibility of time (very familiar and universally accepted) as its generalization. As this is very well known, time is necessarily the one special quantity within Pauli's "particle paradigm" of energy conservation, which is not supplied by a corresponding Hermitian operator, since it is irreversible in definition directly contradicting the inherent reversibility of any Hermitian operator.

So, one can clearly suggest that duality or complementarity originates from the irreversibility of time, meanwhile generalized as the pair of both abstract and mathematical reversibility and irreversibility, if one considers the idea of the "nonstandard bijection" as an implementation of reversibility as two dual and complementary irreversibilities in turn originating from the quantity of time ("as it is": especially discernibly in *classical* quantum mechanics) in the final analysis. Meaning that understanding of the nonstandard bijection, it implies immediately the case where both mappings are continuous, and thus the derivative idea of the nonstandard homeomorphism equivalent to the homeomorphism of a bit of information: and then, applicable in relation to Poincaré's conjecture.

As an additional, but very important conclusion, the homeomorphism of a bit of information (and thus, Poincaré's conjecture) relates directly to the problem about the completeness of mathematics (or to the option to prove its self-consistency within itself and by itself): which is the thesis advocated by Hilbert mathematics. The following statement is justified in detail in other papers (Penchev 2022 June 30; 2021 August 24; 2021 April 12): the fundamental bit of information can very well represent the structure of the relation of propositional logic, (Peano) arithmetic, and (ZFC) set theory; or speaking loosely and rather figuratively, set theory and arithmetic are the same Boolean algebra interpreted differently in relation to arithmetic, which in turn is the "half" of set theory (or respectively, of set theory for both are the same structure of Boolean algebra).

Then, the distinction of propositional logic to set theory in relation to arithmetic can be expressed so: the former is the privileged zero-order logic, to which the former is the class of all possible first-order logics. This means by the mediation of the above "metaphor of the halves" that set theory considers the general case of both "halves", and propositional logic is inherently incapable do differ the case of a single "half" from that of both so that set theory and propositional logic can be formally identified as the same Boolean algebra or as two different interpretations of that structure.

Finally, the nonstandard homeomorphism can substitute the nonstandard bijection in the above relation of arithmetic, set theory, and propositional logic as follows. If one applies the axiom of choice or the equivalent well-ordering "theorem" to any continuous bijection (including the nonstandard one), it would be reduced to an arithmetic bijection which depicts a Peano arithmetic into another or in other words, it supplies both sets equated (in a sense) by the bijection with the structure of Peano arithmetic. So, if the bijection is nonstandard, the corresponding sets are provided by two dual and anti-isometric copies of Peano arithmetic. This means that the homeomorphism of a bit of information implies the option of Fermat arithmetic mentioned already in relation to Fermat's last theorem.

VIII THE FOUR-COLOR THEOREM

As this is very well known, the four-color theorem is proved, but in a unique and maybe even “scandalous” way as far as the software assistance is crucial. The admissibility of that help is not a subject of the present paper, but only a possible “human” rationalization. The available proof enumerates a finite set of all types of any neighborhoods of “countries on the maps”. Then, it infers that all possible cases can be reduced into the finite set of patterns.

This is the first huge advantage of the cited proof since any uncertain finite sets according to the arithmetic axiom of induction as well as any actually infinite sets according to set theory are equally well mappable into a certain finite set such as that one enumerating all ways (being a certain natural number) for any areas to be adjacent on a two-dimensional (Euclidean) plane. The advantage is similar to Aristotle’s ancient approach to overcome the gap and opposition of Plato’s “things versus ideas” and then repeated many times again and again in different forms: including Husserl’s phenomenology to philosophy at all or Russell’s logicism especially to the foundations of mathematics. That mapping into a certain finite set justifies “epoché to infinity” just as propositional logic is able to do this though in an alternative way.

This is an illustration of the program of finitism[20] (sometimes called “strict finitism”[21] or opposed to “inductive finitism”). It overcomes the Gödel dichotomy about the relation of arithmetic to set theory (just as logicism), however generating a problem (maybe even fundamentally insolvable) about whether there always exists a relevant certain finite set as well as whether the last statement can be proved or it is inherently unprovable and should be eventually postulated. After the distinction of Hilbert (non-Gödelian) mathematics versus Gödel (Gödelian) mathematics, one can notice that the argument of finitism is valid in Hilbert mathematics at all and then the hypothesis that a relevant certain set as above exists always if and only if the corresponding statement is provable in Hilbert mathematics can be suggested.

On the contrary, it can be rather easily proved in Gödel (Gödelian) mathematics (as this is above sketched) without the finitist method of enumerating all cases indicating each of them one by one. The suggested idea consists in utilizing the aforementioned homeomorphism of a bit of information as after Poincaré’s conjecture. Indeed, any bit can be represented as two complementary or dual oppositions: “before the choice” versus “after the choice”; “the one alternative after the choice” versus “the other alternative after the choice”.

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[20] Finitism in mathematics, in philosophy of mathematics, or its relation to Hilbert’s program is discussed in many papers (e.g. Hämeen-Anttila 2019; Dean 2018; Ebbs 2016; Magidor 2015; 2007; Incurvati 2015; Ye 2011; Ganea 2010; Sanders 2010; Feferman 2008; Bremer 2007; Haukioja 2005; Kornai 2003; Zach 2003; Mancosu 2001; Suppes 2001; Marion 1999; Levy 1992; Wright 1982; Tait 1981; Webb 1980). Finitism is discussed also ontologically and thus close to Platonism and Pythagoreanism in a series of papers (e.g. Nawar 2015; Puryear 2014; Stenlund 2012; Sanders 2010; Mancosu 2001; Marion 1999; Sharrock, Button 1999; Levy 1992; Tait 1981; Webb 1980).

[21] For example, Dean 2018; Ye 2011; Magidor 2012; 2007; Levy 1992; Wright 1982.

Then, the nonstandard homeomorphism of a bit of information can be interpreted as its projection on two orthogonal axes on the plane, interpretable as the abscissa and ordinate of a Cartesian coordinate system under the condition that its beginning corresponds to the bit at issue. The homeomorphism can be visualized as the identity of any point by itself and then once again, as the beginning of a coordinate system, i.e., as the cross point of its abscissa and ordinate. Two colors are sufficient to distinguish any two adjacent points in each of both axes: or totally four colors since the axes are orthogonal and thus absolutely independent of each other.

Further, the way of how the nonstandard homeomorphism of an information bit implies the four-color theorem (but only in Gödel or Gödelian mathematics) can be demonstrated by *reductio ad absurdum*. Let us admit some map, on which a fifth color is necessary and it is projected on both abscissa and ordinate, after which two options appear. Either the abscissa or the ordinate needs a third color to distinguish two adjacent areas, which is impossible and implies by *modus tollens* that the initial suggestion for any fifth color ostensibly necessary is false. The other option allows for the fifth color to vanish after projecting on both axes. This contradicts the nonstandard homeomorphism of a bit of information directly, though. Translated into terms of quantum mechanics, it corresponds to the absence of hidden variables in it (Neumann 1932; Kochen, Specker 1967) since any fifth or next colors imply them.

The argument by *reductio ad absurdum* has the additional advantage to explain why that proof (but unlike the established one by enumerating the finite number of all cases) is not valid in Hilbert mathematics anyway remaining correct in Gödel or Gödelian mathematics. The area corresponding to any nonzero distance between finiteness and infinity needs a "fifth color" being different from both pairs of colors necessary for both dual and anti-isometric copies of Peano arithmetic (for example as the "zeros" and "units" of each of both binary numbering systems for recording any natural number in each copy).

One can summarize that the four-color theorem is rather similar to Poincaré's conjecture at least as to the relation of Hilbert and Gödel (or Gödelian) mathematics both needing proofs relying on the homomorphism of a bit of information, but the former allows for the "finitist argument" in addition.

Finally, one can notice that four-color theorem, once it has been related to that information bit homeomorphism and thus to Hilbert arithmetic, can be immediately generalized as an alleged "four-letter theorem" stating that any entity whether mental or physical can be named by a minimal four-letter alphabet, to which the four-color theorem is a particular case. The four-letter theorem may seem to be partly counterintuitive:

If one considers the set of all entities, it might be well-ordered by the axiom of choice and a unique number to be assigned to it. Then, the two "colors" of "0" and "1" should be absolutely sufficient for the alphabet of the universe. However, the set of all entities is self-contradictory since it has to contain the set of all sets being in turn self-contradictory after Russell's paradox. Thus, the two-letter alphabet of the universe suffers from the Gödel incompleteness and needs the two additional letters provided by Hilbert arithmetic in the final analysis.

Two other examples being interpreted as particular cases of the four-letter theorem can make it clearer: (1) the teleportation theorem, requiring two additional oppositions (i.e. a single bit after the present consideration, but unlike the usual prejudice identifying a bit with a *single* opposition) to be transmitted by a classical channel obeying the limit of locality, i.e. the speed of light in a vacuum to complement the instantaneously messaged quantum information to be unambiguous; (2) the "four letters" of DNA or RNA, guanine (G), cytosine (C), adenine (A) and thymine (T) (respectively, uracil in RNA) allowing for the hereditary information to be transmitted to the next generation whether through and by cell division or sexual reproduction.

The latter interpretation hints at a different understanding of the pair of finiteness and infinity as finiteness doubled by its dual (and anti-isometric in the case of Hilbert arithmetic) counterpart. That doubling can be identified as biological reproduction and then generalized as reproduction at all. While cell division suggests rather literally copying (though obeying random mutations in general) and thus corresponding to Gödelian mathematics, its evolved counterpart of sexual reproduction suggests some degree of "entanglement" (but limited to a single biological species or sufficiently related species) since the male and female DNA do not coincide absolutely. In other words, the description by non-Gödelian mathematics would be relevant in the latter case.

As to the teleportation theorem, it can be immediately interpreted as the distinction of locality and nonlocality physically or that of infinitesimality (whether "small" or "great") and finiteness mathematically. One can imagine an arithmetically inductive process (corresponding to a complete actually infinite set) of transmitting the same qubit (of quantum information) from point to another in spacetime, needing an additional bit through a classical channel thus obeying the limit of the speed of light in a vacuum. Then, one can identify the transmitted nonlocal qubit with the infinite string of bits by the process of touring all spacetime points, after which it would be doubled by its dual counterpart belonging to the dual qubit Hilbert space.

IX THE YANG-MILLS EXISTENCE AND MASS GAP PROBLEM

In a sense, the Yang-Mills existence and mass gap problem justifies Hilbert mathematics in a way sketched in more detail above; as well as vice versa: Hilbert mathematics implies its solution so that the positive solution is attached to any finite nonzero "distance between infinity and finiteness", and the negative one corresponds to Gödelian mathematics. Accordingly, the Yang-Mills existence and mass gap problem seems to be the serial Gödel insoluble statement in Gödel mathematics since even its formulation refers to the complement of arithmetic to set theory being inherently ambiguous.

The present section will consider the above statements one by one starting by the second claim: how Hilbert mathematics implies the solution of the problem. The "simple compact gauge group" can be "deciphered" by terms of the relation of the infinitesimally small to the finite mathematically (respectively, the finite to the infinitesimally great) or by that of the local to the nonlocal physically. This means only a translation into "Hamiltonian" (language), after which the relation at issue is transformed into a functional group, or particularly, into a parametric group where the parameter is the quantity of calibration.

Then, a few cases can be distinguished to the pair of the initial variable (physically interpretable as the measured quantity "by itself") and the functional variable (physically interpretable as the quantity represented by the readings of the apparatus); they can be: (1) gapped (physically implementable by the qualitative gap between any two different physical quantities including the most important particular case of conjugate quantities); (2) overlapped (i.e., the general former case is reduced only to conjugate quantities where the physical dimension of the overlapping is the product of the physical dimensions of any pair of conjugate quantities); (3) neither gapping not overlapping: the input and output variables pass smoothly into each other, i.e., they are the same quality: what the standard (i.e. Gödel) mathematics postulates though implicitly.

The former two cases can be furthermore unified and opposed to the third one, and the overlapping dimension may be postulated as the quantity of action as physics does. Then, one is enough to follow the pattern thoroughly elaborated by quantum mechanics and information (sometimes denoted as "quantum *informatics*" for emphasizing to be a *science*, i.e., an area of human cognition), after which the limit of the minimal overlapping indicated by the Planck constant implies quantum discreteness and leaps, on the one hand, and probability (density or not) distributions equivalent to wave functions whether qubit or not, on the other hand.

The former "hand" implies any finite nonzero mass gap for the fundamental constant of the speed of light in a vacuum, if one postulates any time unit "t_0": $m_0 = \frac{h}{c^2 t_0}$. Finally, the arbitrary convention for "t_0" can be overcome by the third fundamental constant, the gravitational constant so the mass gap "m_0" to be identified as the Planck mass: $m_0 = m_{Pl}$. So, any finite nonzero overlapping, which is somehow chosen to be the Planck constant in our universe, implies unambiguously a certain mass gap once the real mathematics in the framework of Hilbert mathematics is granted to be non-Gödelian.

The latter "hand" implies the Yang-Mills existence for any non-Gödelian Hilbert mathematics (thus establishing an equivalence of the both "halves" of the problem as long finiteness and infinity can be correlated by any nonzero parameter)[22]. Indeed, any non-Gödelian Hilbert mathematics means a finite nonzero distance between finiteness and infinity, which can be granted to be always positive ("overlapping") without restricting the statement, and thus discreteness, a gap between the dimensions of the input and output variables, which can be further interpreted always as the quantum complementarity of conjugate quantities such that the physical dimension of their product is action:

A corollary is that the entire simple compact gauge group acquires discreteness and respectively probabilistic uncertainty rather than the limited area of the distance ("overlapping") at issue, by means of which it is doubled by a dual counterpart as the pattern of Hilbert arithmetic where Hilbert arithmetic in a narrow sense is doubled by the qubit Hilbert space. The difference consists in the fact that the doubling is inferred in the former case, but postulated in the latter where and since it is consistent with both Gödelian and non-Gödelian Hilbert mathematics.

[22] However, one can additionally admit the Yang-Mills existence even in Gödelian or Gödel mathematics, which will be discussed in more detail a little below in the present section.

That doubling is equivalent to a Yang-Mills theory and the Yang-Mills existence. The doubling at issue can be deduced from any finite nonzero mass, but it is consistent also with zero mass gap because it can be interpreted as an infinitesimally small mass gap reducible to the former case. The last observation is due to the specific peculiarity of the gauge group (at least if it is simple and compact) to be interpretable as a derivative to the mapping of an infinitesimally small (great) set into a finite set or vice versa. Then, the overlapping area implies for the mapping at issue to be probabilistic in order to avoid the direct logical contradiction as to the overlapping: both an infinitesimally small (great) and finite quantity simultaneously.

The overlapping region can be interpreted by another version of "Schrödinger's cat", restoring the initial situation of the measurement of a quantum entity by a macroscopic apparatus, to which the "feline" allegory refers: the coherent quantum state of some radioactive atom (i.e., representable as a qubit) is mapped into the corresponding dead-and-alive state (i.e., a bit of information). In other words, the local space of states of the atom is "gauged" by the global state of the cat, or "calibrates" it.

So, even the overlapping area is granted to be infinitesimally small thus effectively resulting in an "almost flat" Gödelian mathematics, which is sufficient for Gödelian mathematics to be consistent with the Yang-Mills existence. This is easy to be illustrated by overlapping the areas of the infinitesimally small and the finite in a "Hamiltonian" (language), in which they are inherently distinguished as two variables independent of each other. The overlapping area can be also visualized as containing Gödel insoluble statements, which can be anyway resolved (after rejecting the "contradiction option" of the dichotomy and following the "incompleteness one") just by complementing with the Yang-Mills existence in the final analysis.

Also vice versa: one may see the Yang-Mills existence and mass gap problem as the serial one due the dichotomy about the relation of arithmetic to set theory in Gödel mathematics since it turns out to be an insoluble statement because of the following reason. It needs duality to be resolved whether positively or negatively. On the contrary: without involving duality, it is absolutely uncertain, ambiguous, really a Gödel insoluble statement.

X A HYPOTHESIS INSTEAD OF CONCLUSION

The conjecture is the following. Many, if not almost all of the most difficult problems of contemporary mathematics rely on (at least or touch) the Gödel dichotomy about the relation of arithmetic to set theory: either incompleteness or contradiction. In other words, and following terms of the present paper, they are insoluble in Gödel mathematics and thus needing the new dimension of Hilbert mathematics just consisting in the parametrization of that relation as a real variable.

The metaphor about how to build four triangles by six segments as needing the "new" third dimension to the two ones of a plane might be again utilized. No technical mathematical skills or intellect can replace creativity for insight, a new viewpoint to the problem, after which it turns out to be technically and mathematically elementary even trivially. Even a "feeling of deception" can appear: some "trick of a conjurer" somehow diverted our attention while performing unobserved manipulations, which led to the illusion that the problem was supposedly solved:

Not at all! The solutions are real though being despairingly simple in Hilbert mathematics.

The impossibility of being resolved is seeming, ostensible, alleged, and due only to the prejudice omnipresent in our age that mathematics and the physical world are opposed. What philosophy of mathematics "dares" at the best is … Platonism. Pythagoreanism claiming for mathematics to be "first philosophy" and ontology is nonsense nowadays or after the same prejudice. Mathematics is able only to create models partly fitting reality according to it: moreover, the antithesis seems to be so madcap that even it cannot be discussed seriously.

The present paper demonstrates something quite different and rather similar to Lobachevski's amazement trying to prove Euclid's Fifth postulate by the method of *reductio ad absurdum*, but shockingly establishing that no contradiction appears at all. He built a new geometry other than Euclidean one, but nonetheless also consistent. So, one might follow Euclid admitting the negation of the prejudice at issue and then, searching for any contradiction to appear. Alas, no contradiction appears just as it did not after Lobachevski's enterprise. The prejudice at issue is not some higher wisdom, but only an occasional human convention and thus not better than its logical negation.

However, if one utilizes the analogy (which is more than a superficial analogy) to non-Euclidean geometry, the approach of the present paper is that of Riemann rather than Lobachevsky's: culminated into pseudo-Riemannian space and Einstein's general relativity utilizing it as a "mathematical model" somehow turning out to be relevant to physical reality; that is, just according to the prejudice at issue. However, the present paper does not reckon any more pseudo-Riemannian space as a model of a reality, but as *reality itself*, in the framework of which physical reality appears as a particular case by virtue of only mathematical laws and necessity.

One can add the following tenet in favor of the hypothesis. Too many mathematicians, very smart and professional, for more than four centuries in the case of Fermat's last theorem as an example, have been trying to resolve them as proper mathematical problems as they are formulated. However, they could not manage since the prejudice at issue is *metamathematical* or *philosophical* referring to the position of mathematics in the organization of cognition in Modernity.

Though many mathematicians or physicists were philosophers in Descartes and Leibniz's age, they are now opposed in the beginning of the 21th century. "Shut up and calculate!" (a slogan maybe misattributed to Richard Feynman) anyway expresses very well the essence of classical quantum mechanics after Pauli's "particle paradigm" of energy conservation as well as the anti-metaphysical "milieu" of natural science and mathematics at all nowadays. Philosophy is restricted to human and social problems, for example, such as gender and social equality, etc., but opposed to "metaphysics" stigmatized to be ostensibly anti-scientific. The real scientist should not, ought not to (or even dare not) be a philosopher.

Indeed, Einstein and Bohr, or Russell and Brouwer were philosophers not less than scientists, but an anti-metaphysical "coup d'état" was accomplished in the middle of the 20th century, after which for physicists and mathematicians to be philosophers became shameful and indecent: the motto "Shut up and calculate!" regardless of its authorship or authenticity represent that anti-philosophical "milieu" of science now. So, the "great and unresolvable mathematical" problems (such as the seven CMI Millennium ones) are not proper mathematical in fact, but rather philosophical. They are great mathematical ones only in that "milieu" imposing censorship and self-censorship to any, even timid attempt to philosophize:

Those statements are really exceptionally difficult puzzles only after that kind of intellectual dictatorship and obscurantism, which is completely forbidden to criticize and which is glorified as an "age of reason", just as every totalitarianism requires to be spoken only the *exact opposite of what it is.* On the contrary, their demystifying to be rather elementary or even trivial after a relevant "Gestalt change", adding a new, both philosophical and mathematical, dimension as Hilbert mathematics does, demonstrate visually that the intellectual totalitarianism, censorship and self-censorship, dictatorship and obscurantism are really intellectual totalitarianism, censorship and self-censorship, dictatorship and obscurantism. "The King is naked!" rather than dressed in very beautiful and expensive clothes of very difficult and unresolvable puzzles accessible only to tailors' great minds (such as that of Andrew Wiles).

This is false! The King is really nude! Those of the great problems which are resolved show that they are really insoluble after the dogmas of Gödel mathematics, and the insolubility at issue vanishes in thin air after the correct viewpoint to them. However, their solution in the circumstance of intellectual obscurantism, totalitarianism, censorship and dictatorship is so sophisticated, a so complicated allegorical "Aesopian language" is necessary to be described solutions forced to transcend the prejudice but represented in false "clothes" as if confirming it again and again as any totalitarian propaganda requires from any opinion or statement. So, Andrew Wiles cannot but dress its proof in almost all contemporary mathematics only to hide the simple fact that it contradicts the prejudice, and merely recognizing this, it can be reduced to a few pages (for example, in an explicit inverse Grothendieck universe, which is unavoidable in the final analysis).

So, the conjecture can be paraphrased more or less aphoristically: the most or even almost all of the most difficult mathematical problems are so insoluble because a part of their solutions is beyond mathematics, and within philosophy.

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