Variations on Menger-Diaz Sponges

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Abstract

Fractal geometry is a branch of mathematics that deals with the study of patterns that repeat themselves infinitely in different scales. In this article, we propose a method to expand upon Menger-type fractal constructions to generate a variety of designs called Menger-Diaz fractals. The proposed method allows for the manipulation of the initial "atoms," rules, and distances to create a range of intriguing figures, including cubes or polyhedral shapes. We also describe how to apply this process to other polyhedra besides cubes or combinations of compatible polyhedra. Additionally, we investigate the concept of a recursive "atom" that is fundamental to the Menger process and can be a cube, a tetrahedron, or any other polyhedron that tessellates space. We present the most outstanding never seen before figures and the fractal dimension and volume of all them.

Introduction

The Sierpinski tetrahedron is a fractal geometric shape that is named after the Polish mathematician Wacław Sierpiński. It is a three-dimensional analogue of the Sierpinski triangle, which is a fractal made of repeated iterations of removing triangles from a larger triangle, in a Menger-like style.

To create the Sierpinski tetrahedron, you start with a regular tetrahedron (a pyramid with a triangular base), and then repeatedly divide each of the four triangular faces into smaller triangles by connecting the midpoints of each side. Then, you remove the central tetrahedron that is formed by connecting the midpoints of the four faces of the original tetrahedron. This process is repeated on the remaining tetrahedra, ad infinitum.

As the process continues, the resulting shape becomes increasingly complex and resembles a sponge-like structure with an infinite number of interconnected cavities. It has a fractal dimension of approximately 2.7, which means it is more complex than a two-dimensional shape but less complex than a three-dimensional solid.

In a recent submission to the Bridges conference, the authors have described how to expand upon Menger-type fractal constructions to generate a variety of designs[1]. The authors propose a method that allows for the manipulation of the initial "atoms," rules, and distances to create a range of intriguing figures. Additionally, this method can be utilized to produce a new variant of Menger fractals called Menger-Diaz, achieved by modifying the process of removing initial "atoms" at each level. This modification results in a distinctive shape with distinct characteristics, demonstrated through various examples.

To create the cubes or polyhedral Menger-Diaz we start with a cube of edge length 1, which we iterate by dividing it into smaller cubes with edge length 1/n and discarding d of them in appropriate positions. We repeat the process with the remaining n^3-d smaller cubes. The resulting portion of the cube after infinitely many iterations is called a sponge and can be obtained using various rules of discarding. We can apply this process to other polyhedra besides cubes or combinations of compatible polyhedra.

Imagine a cube or prism with sides of length n, consisting of 1's and 0's. We begin by designing a face with holes, which we then duplicate in the final row, ensuring that all intermediate layers have transparent zeros. The second and second-to-last columns should resemble the first face. The concept of a recursive "atom" is fundamental, as it is something that fits with its copies to build a new atom in each iteration of the Menger process. This atom can be a cube, a tetrahedron, or any other polyhedron that tessellates space. The process remains the same regardless of the chosen shape. There are many possible cubes with 1's and 0's, but we are only interested in the symmetrical ones with clear holes, the symmetrical ones, or the dual ones.

Discussion

Imagine a cube of side n (or a prism of rectangular faces) de 1 and 0. First we define one face. With each face and his axe in the center of the face, the figure that we search must be invariant to the rotation of 90^0 of each axe. Then we search that the figure resultant can be clear in his 0, we can see the other side with a light. Of other form, all the faces of the cube must be equals and vacuum must be face to face. Cubes with 1 and 0 there are many possible 2^{nxnxn} (too many, more than the number of grains of sand in the Earth) but we only are interested, here, by the “regular ones”, symmetrical and with the clear holes, or only symmetrical, or the ”dual” ones (The dual rule changes from its original rule that 0 becomes 1 and 1 becomes 0).

The Sierpinski tetrahedron fractal is a case of 2x2x2 Menger-Diaz fractal. There exist various algorithms to generate the Sierpinski tetrahedron, including the one proposed by Alsina and Nielsen in their book[2]: *For example, the first three iterations of the regular tetrahedron and octahedron are shown in Figure7.3.3. In the initial step with the tetrahedron we remove an octahedron leaving four tetrahedra with half the side length of the original and joined at common vertices. In the initial step with the octahedron, we remove eight tetrahedra leaving six octahedra with half the edge length of the original and joined at common edges. The limiting fractals are known as the Sierpinski ´ tetrahedron and the Sierpinski octahedron, since the triangular faces of each are Sierpinski triangles.*

Our algorithm does not distinguish tetrahedron of octahedron. Only in the first “atom”. Three steps: choose an “atom”; choose the distance between atoms in each iteration; choose a cubic set of numbers 1 or 0 indicating the existence or not of atom in this position. We have the nested list of vectors 2x2x2:

a= [[[1,0], [0,1]], [[0,1], [1,0]]]

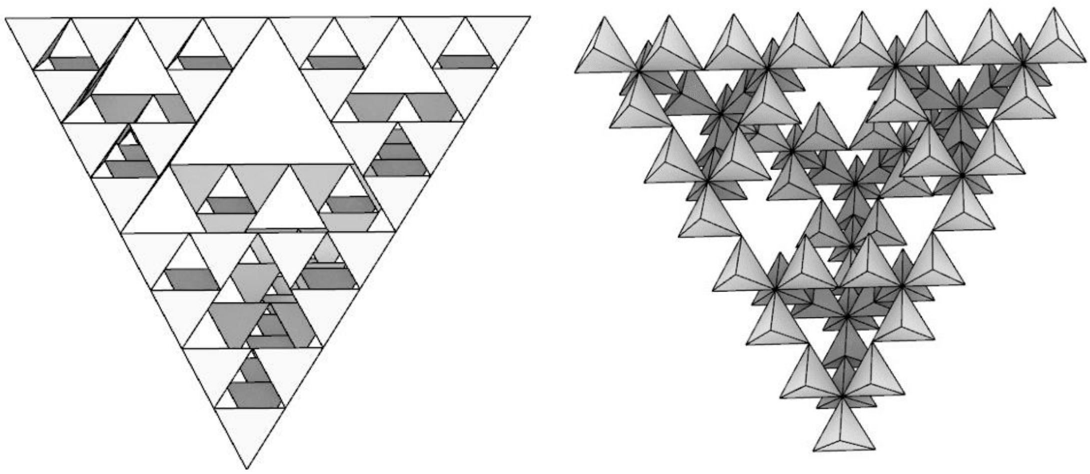

Figure 1: Tetrahedron Menger-Diaz fractal

Only in the tetrahedron the figure dual is different. In the other cases the dual has different orientation. To see that's a Menger-Diaz fractal (because to get it we use the same algorithm) we can change to a cube and obtain that with the cube also gives a similar figure.

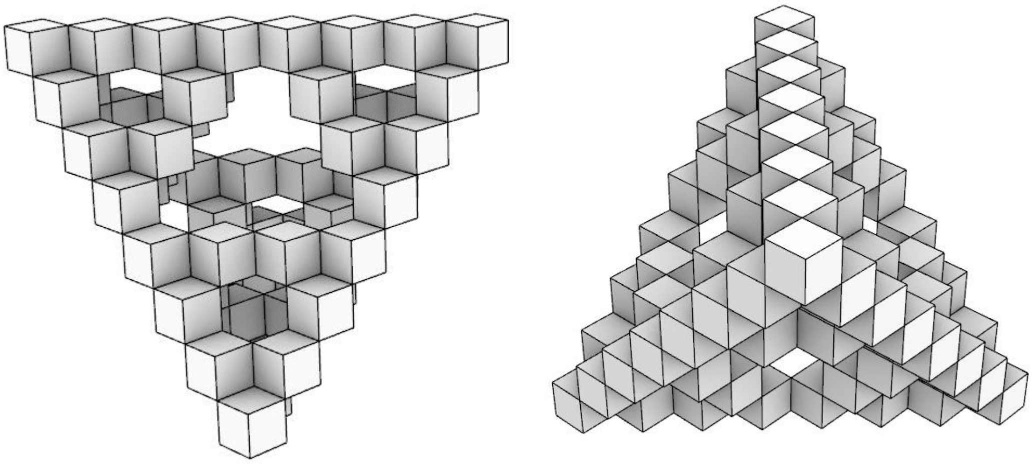

Figure 2: Cube Menger-Diaz fractal

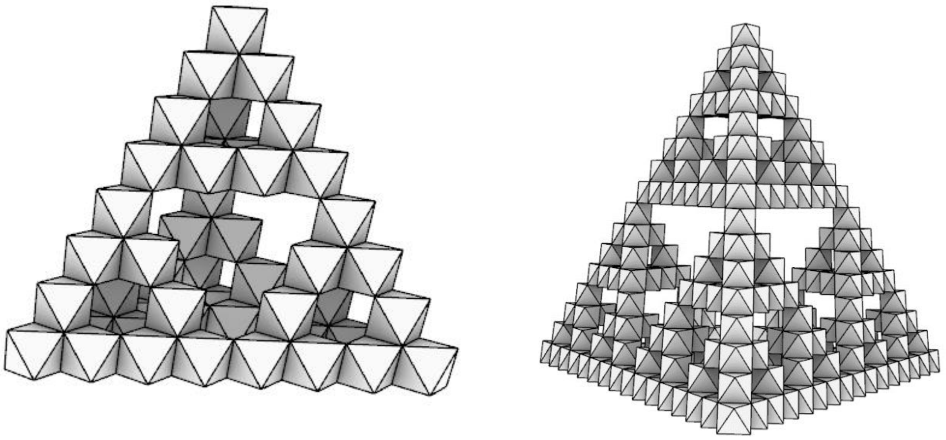

Figure 3: Octahedron Menger-Diaz fractal

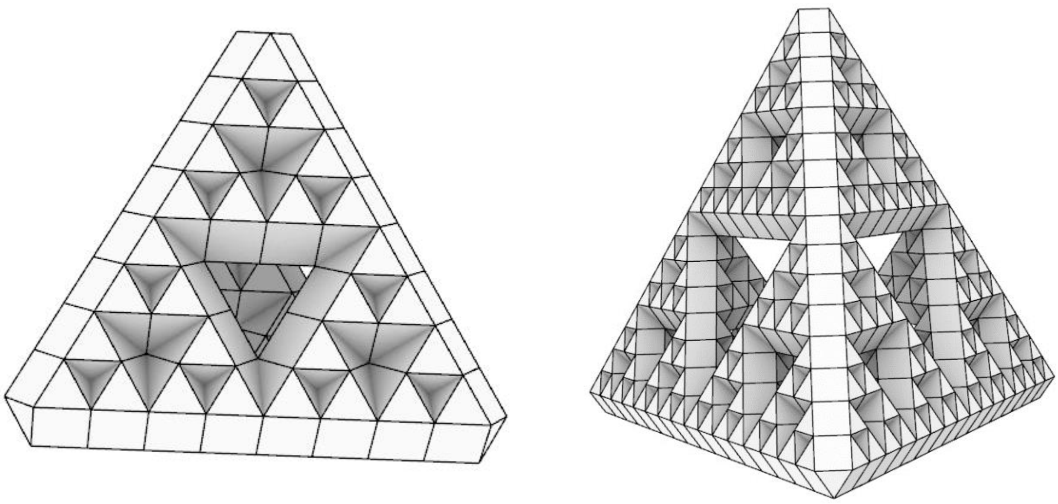

Figure 4: Cuboctahedron Menger-Diaz fractal

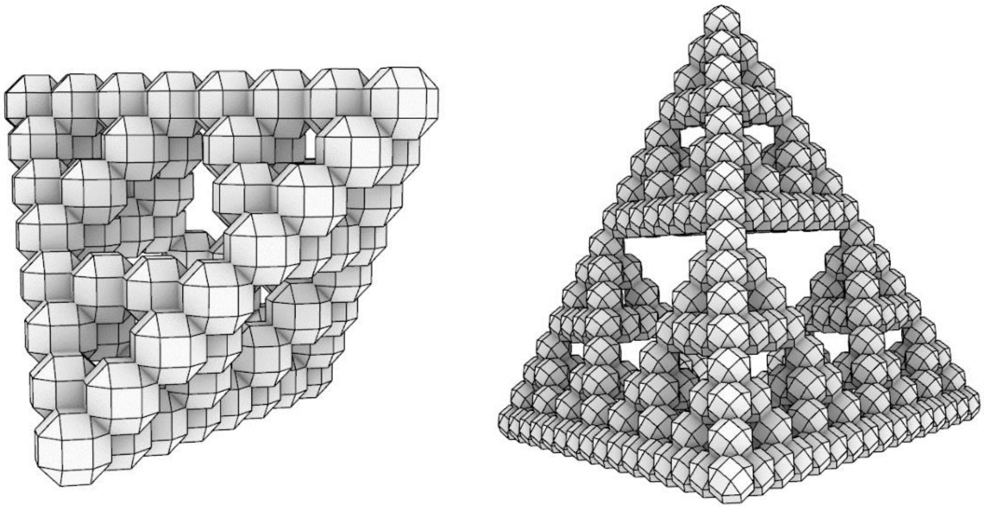

Figure 5: Rhombicuboctahedron Menger-Diaz fractal

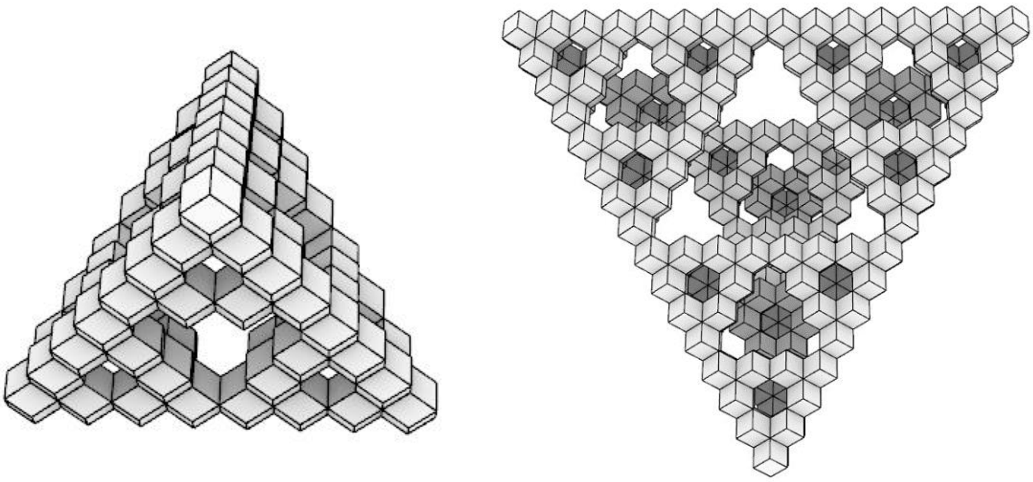

Figure 6: Rhombidodecahedron Menger-Diaz fractal

The Menger-Diaz algorithm offers greater flexibility and applicability, and also has a dual counterpart.

This section presents numerical definitions and images using cubic rules ranging from 4x4x4 to 7x7x7. Since the 3x3x3 rules were studied in the previous article.

a=[[[1,1,1,1],[1,0,0,1],[1,0,0,1],[1,1,1,1]],[[1,0,0,1],[0,0,0,0],[0,0,0,0],[1,0,0,1]],[[1,0,0,1],[0,0,0,],[0,0,0,0],[1,0,0,1]],[[1,1,1,1],[1,0,0,1],[1,0,0,1],[1,1,1,1]]]

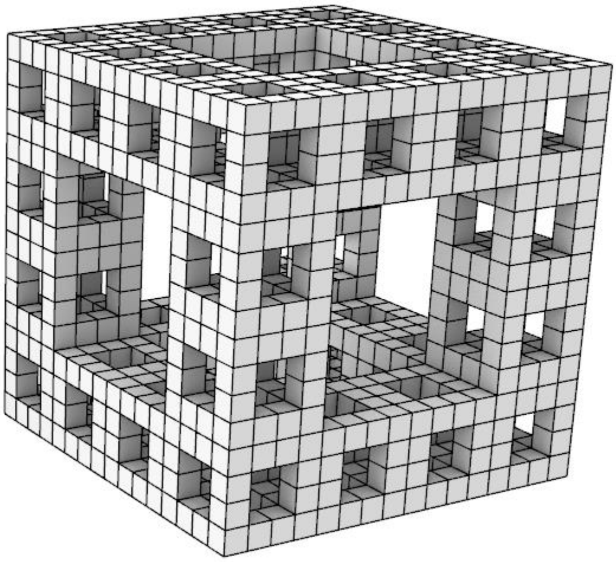

Figure 7: Fractal with great hole.

Another one:

a=[[[0,1,1,0],[1,1,1,1],[1,1,1,1],[0,1,1,0]],[[1,1,1,1],[1,1,1,1],[1,1,1,1],[1,1,1,1]],[[1,1,1,1],[1,1,1,1],[1,1,1,1],[1,1,1,1]],[[0,1,1,0],[1,1,1,1],[1,1,1,1],[0,1,1,0]]]

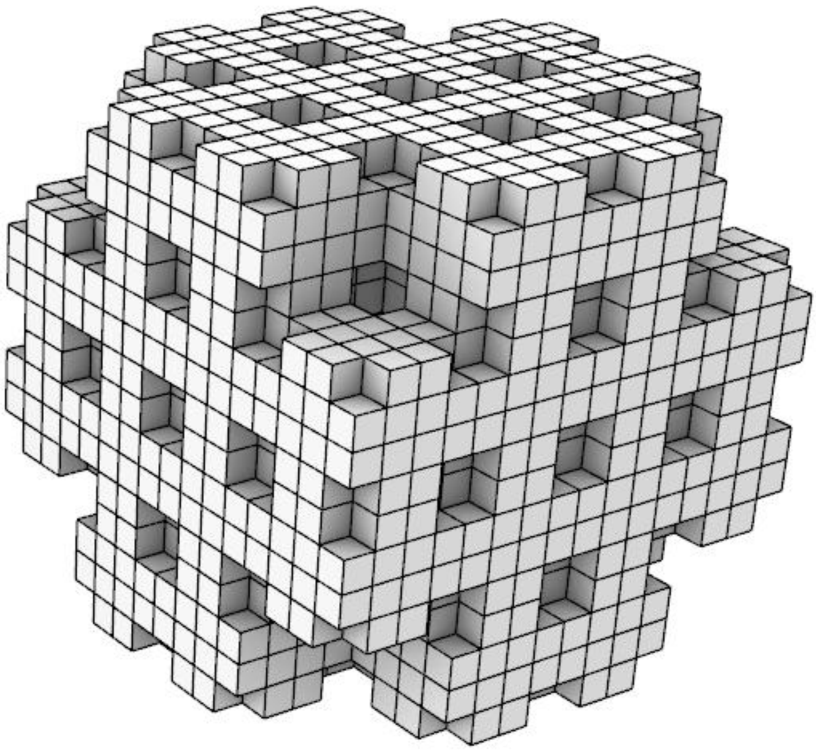

Figure 8: Fractal Double Cross

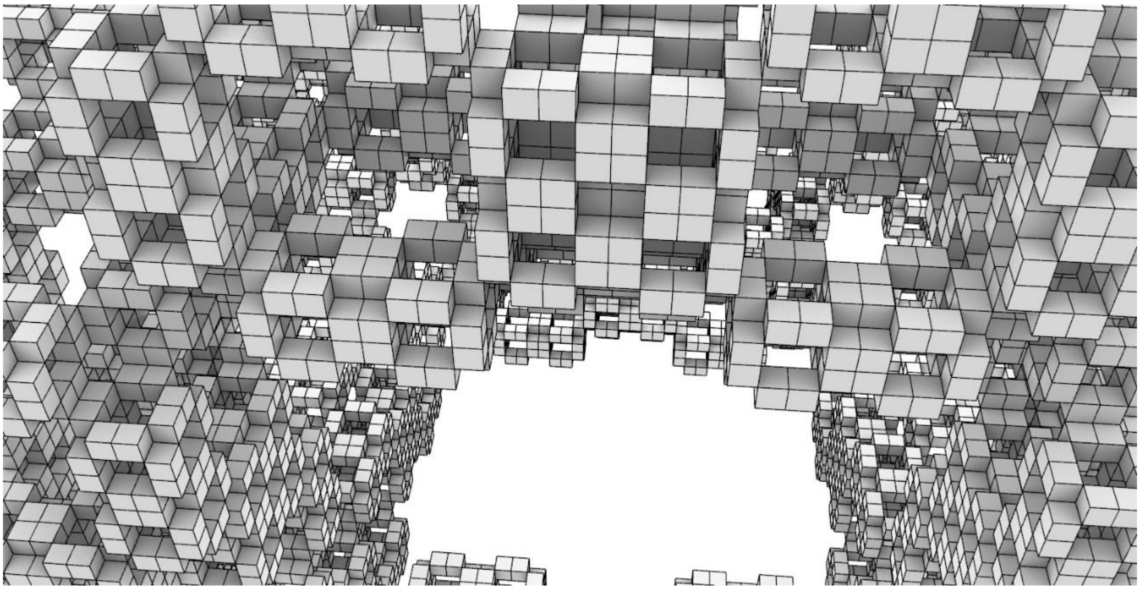

Figure 9: The previous with 4 holes more.

a=[[[1,1,1,1],[1,0,1,1],[1,1,0,1],[1,1,1,1]],[[1,0,1,1],[1,0,1,1],[0,0,0,0],[1,0,1,1]],[[1,1,0,1],[0,0,0,0],[1,1,0,1],[1,1,0,1]],[[1,1,1,1],[1,0,1,1],[1,1,0,1],[1,1,1,1]]]

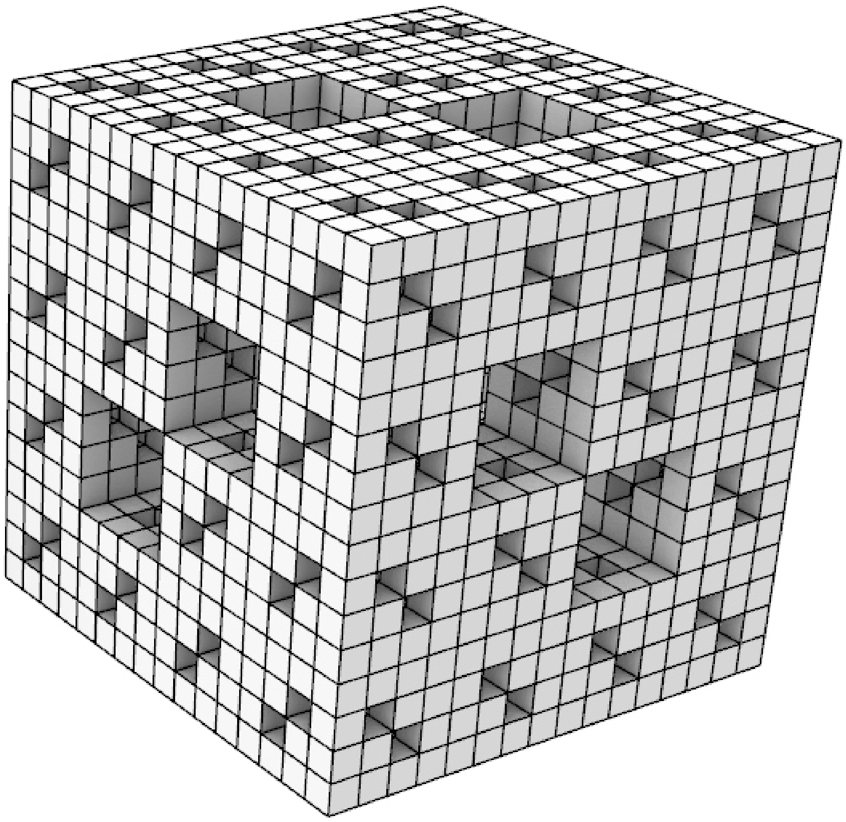

Figure 10: Fractal with 2 holes in the center

This is an example of rule 4x4x4 that we obtain from rules 2x2x2

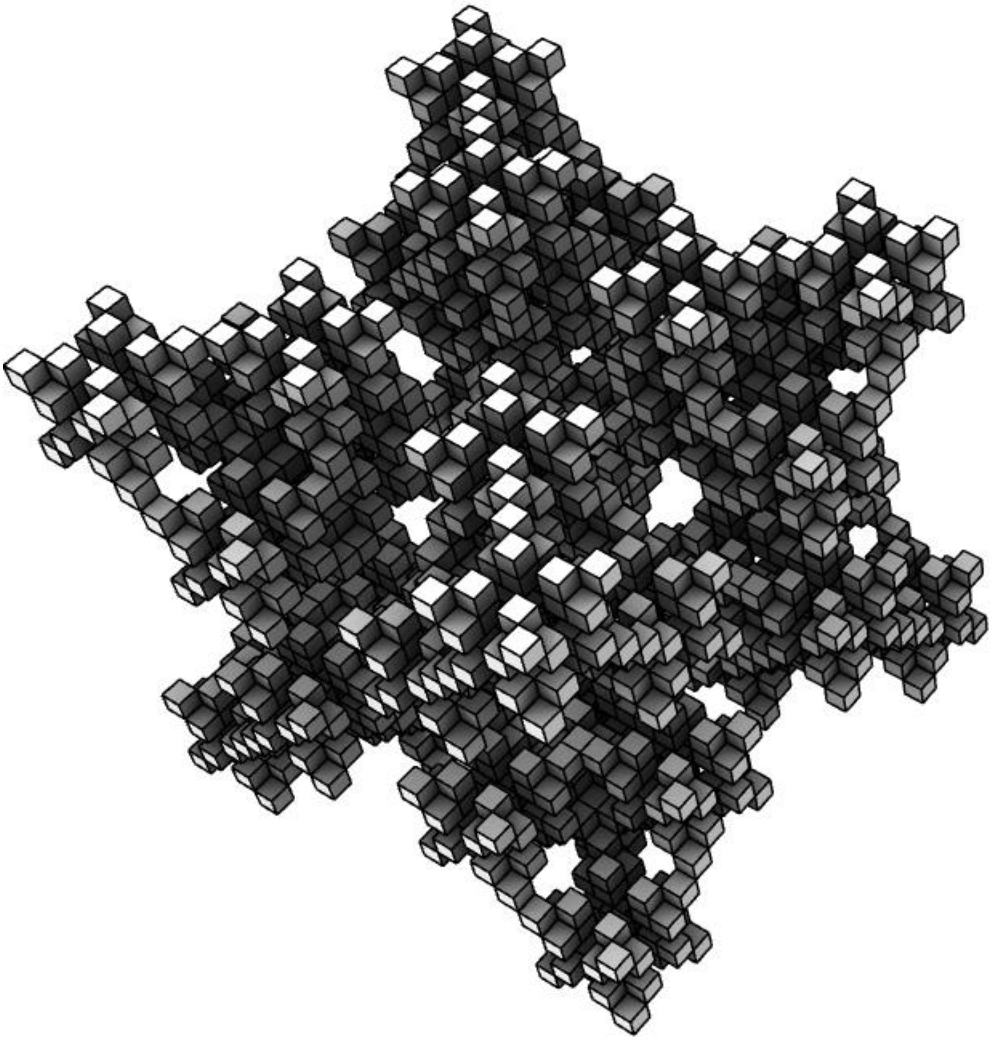

Figure 11: 4x4x4 rule that is the result of two 2x2x2 rules.

The following are examples by following 5x5x5 rules, for:

a=[[[1,1,1,1,1],[1,1,0,1,1],[1,0,0,0,1],[1,1,0,1,1],[1,1,1,1,1]],[[1,1,0,1,1],[1,1,0,1,1],[0,0,0,0,0],[1,1,0,1,1],[1,1,0,1,1]],[[1,0,0,0,1],[0,0,0,0,0],[0,0,0,0,0],[0,0,0,0,0],[1,0,0,0,1]],[[1,1,0,1,1],[1,1,0,1,1],[0,0,0,0,0],[1,1,0,1,1],[1,1,0,1,1]],[[1,1,1,1,1],[1,1,0,1,1],[1,0,0,0,1],[1,1,0,1,1],[1,1,1,1,1]]]

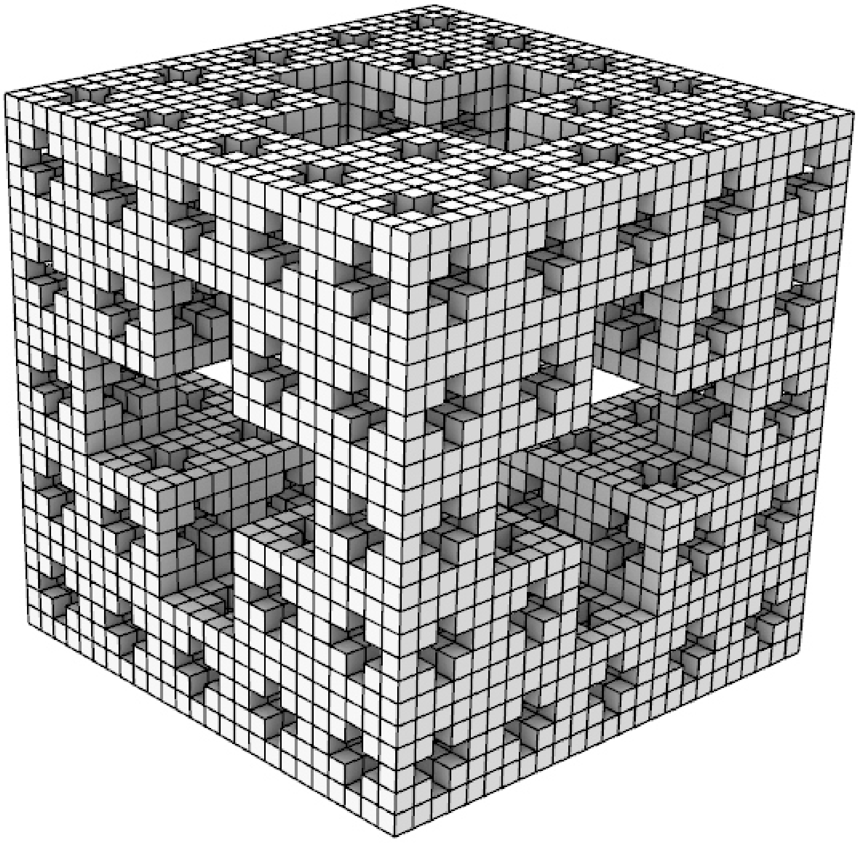

Figure 12: Cross in center

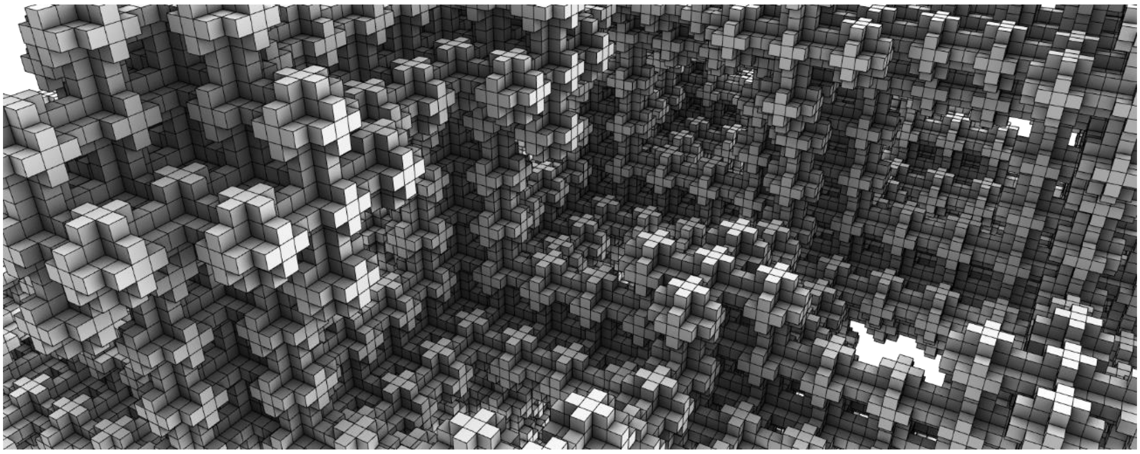

Figure 13: Cross-type dual Menger-Diaz fractal

a=[[[1,1,1,1,1],[1,0,1,0,1],[1,1,0,1,1],[1,0,1,0,1],[1,1,1,1,1]],
[[1,0,1,0,1],[0,0,0,0,0],[1,0,0,0,1],[0,0,1,0,0],[1,0,1,0,1]],

[[1,1,0,1,1],[1,0,0,0,1],[0,0,0,0,0],[1,0,0,0,1],[1,1,0,1,1]],

[[1,0,1,0,1],[0,0,0,0,0],[1,0,0,0,1],[0,0,1,0,0],[1,0,1,0,1]],

[[1,1,1,1,1],[1,0,1,0,1],[1,1,0,1,1],[1,0,1,0,1],[1,1,1,1,1]]]

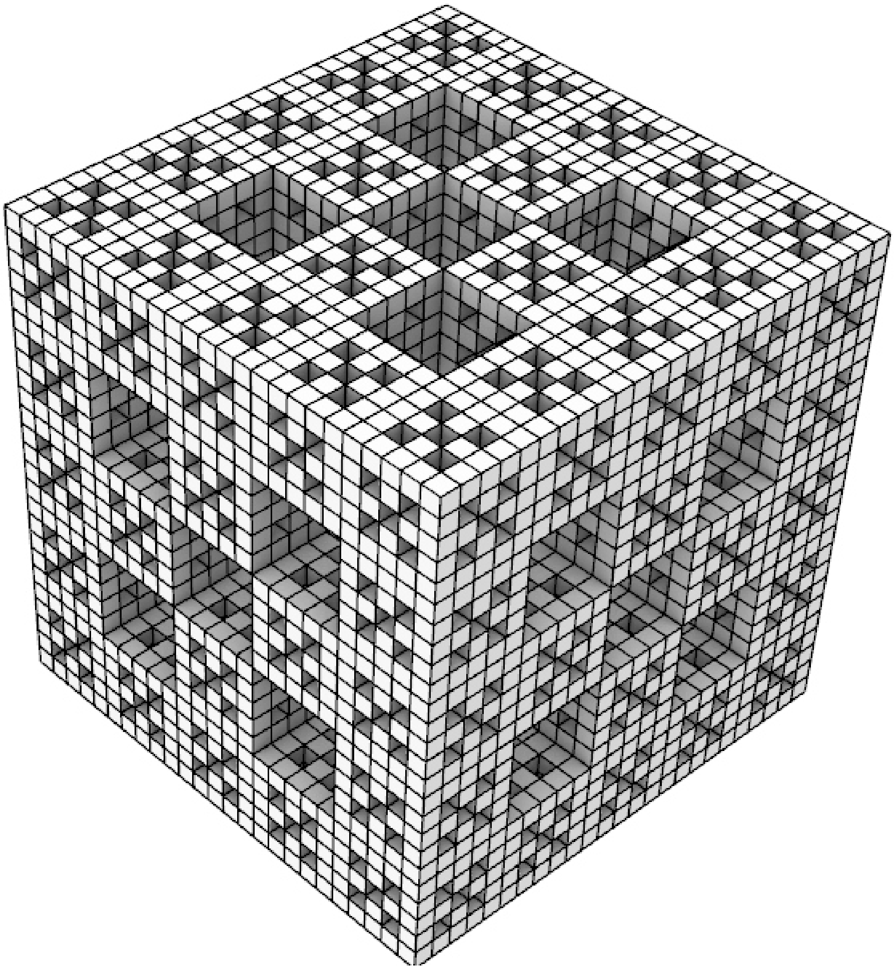

Figure 14: 5 on center or dice of 5

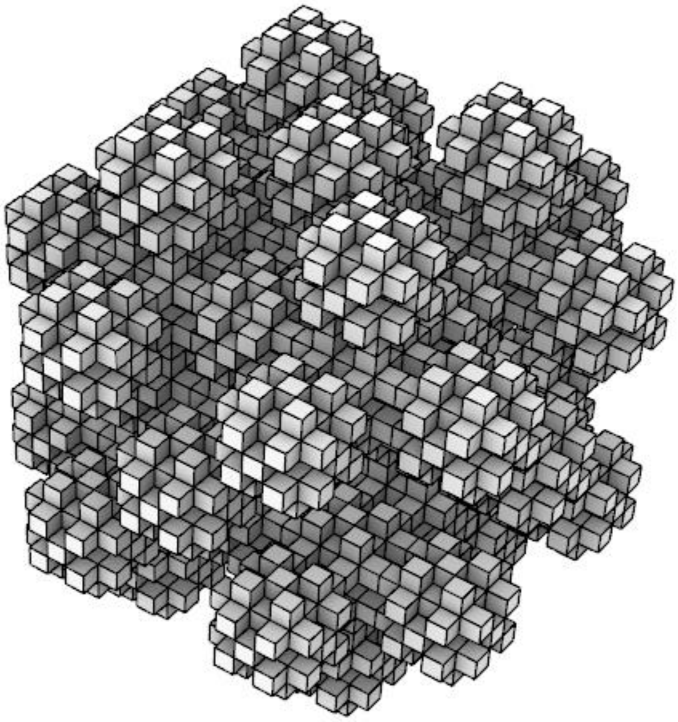

Figure 15: Dual of the previous

Examples of rules 6x6x6

a=[[[0,0,0,0,0,0],[0,0,1,1,0,0],[0,1,1,1,1,0],[0,1,1,1,1,0],[0,0,1,1,0,0],[0,0,0,0,0,0]],[[0,0,1,1,0,0],[0,1,1,1,1,0],[1,1,1,1,1,1],[1,1,1,1,1,1],[0,0,1,1,0,0],[0,0,1,1,0,0]],[[0,1,1,1,1,0],[1,1,1,1,1,1],[1,1,1,1,1,1],[1,1,1,1,1,1],[1,1,1,1,1,1],[0,1,1,1,1,0]],[[0,1,1,1,1,0],[1,1,1,1,1,1],[1,1,1,1,1,1],[1,1,1,1,1,1],[1,1,1,1,1,1],[0,1,1,1,1,0]],[[0,0,1,1,0,0],[0,1,1,1,1,0],[1,1,1,1,1,1],[1,1,1,1,1,1],[0,1,1,1,1,0],[0,0,1,1,0,0]],[[0,0,0,0,0,0],[0,0,1,1,0,0],[0,1,1,1,1,0],[0,1,1,1,1,0],[0,0,1,1,0,0],[0,0,0,0,0,0]]]

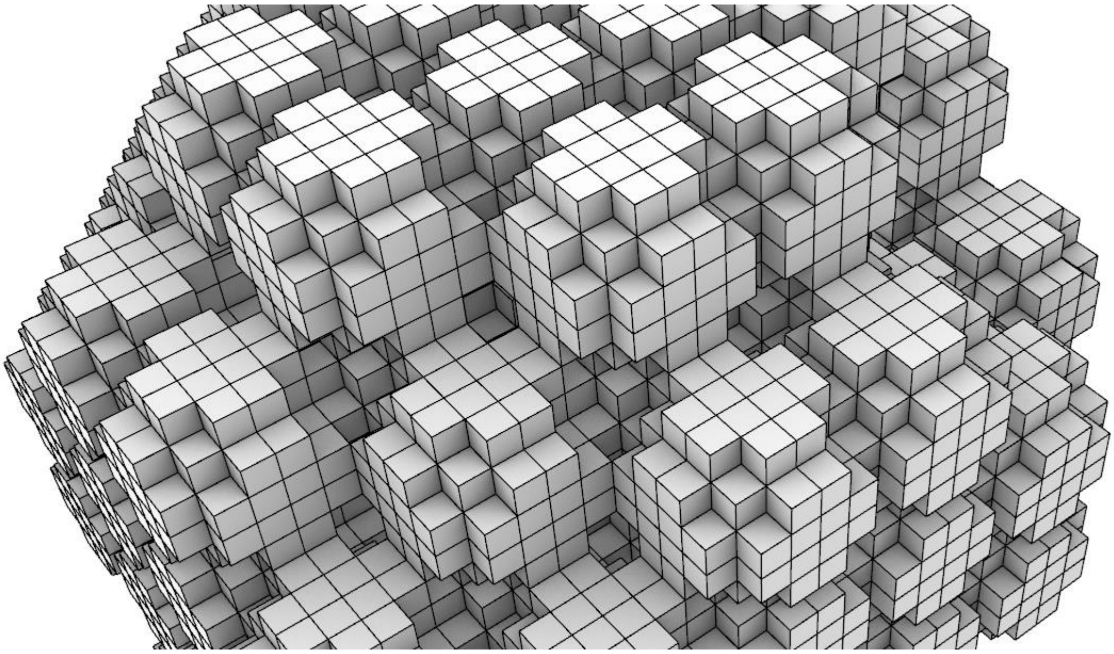

Figure 16: Double cross 6x6x6

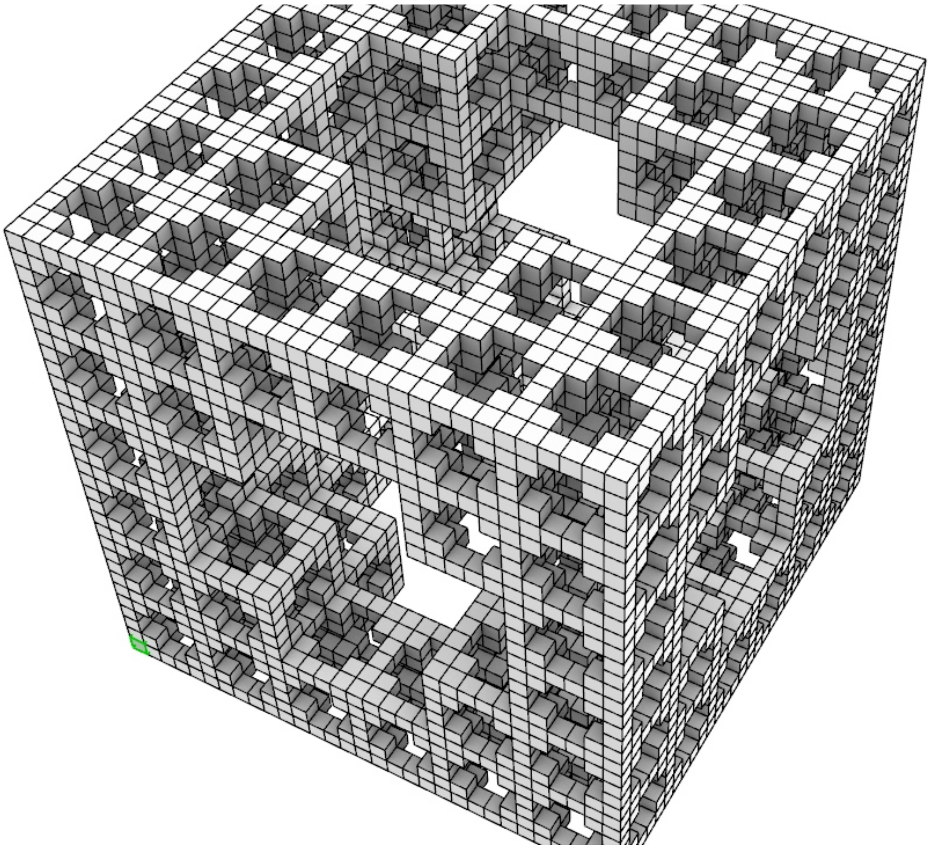

Figure 17: Dual of the previous one

Example of rule 7x7x7

Definition

a=[[[1,1,1,1,1,1,1],[1,0,1,0,1,0,1],[1,1,0,1,0,1,1],[1,0,1,0,1,0,1],[1,1,0,1,0,1,1],[1,0,1,0,1,0,1],[1,1,1,1,1,1,1]],[[1,0,1,0,1,0,1],[0,0,0,0,0,0,0],[1,0,0,0,0,0,1],[0,0,0,0,0,0,0],[1,0,0,0,0,0,1],[0,0,0,0,0,0,0],[1,0,1,0,1,0,1]],[[1,1,0,1,0,1,1],[1,0,0,0,0,0,1],[0,0,0,0,0,0,0],[1,0,0,0,0,0,1],[0,0,0,0,0,0,0],[1,0,0,0,0,0,1],[1,1,0,1,0,1,1]],[[1,0,1,0,1,0,1],[0,0,0,0,0,0,0],[1,0,0,0,0,0,1],[0,0,0,0,0,0,0],[1,0,0,0,0,0,1],[0,0,0,0,0,0,0],[1,0,1,0,1,0,1]],[[1,1,0,1,0,1,1],[1,0,0,0,0,0,1],[0,0,0,0,0,0,0],[1,0,0,0,0,0,1],[0,0,0,0,0,0,0],[1,0,0,0,0,0,1],[1,1,0,1,0,1,1]],[[1,0,1,0,1,0,1],[0,0,0,0,0,0,0],[1,0,0,0,0,0,1],[0,0,0,0,0,0,0],[1,0,0,0,0,0,1],[0,0,0,0,0,0,0],[1,0,1,0,1,0,1]],[[1,1,1,1,1,1,1],[1,0,1,0,1,0,1],[1,1,0,1,0,1,1],[1,0,1,0,1,0,1],[1,1,0,1,0,1,1],[1,0,1,0,1,0,1],[1,1,1,1,1,1,1]]]

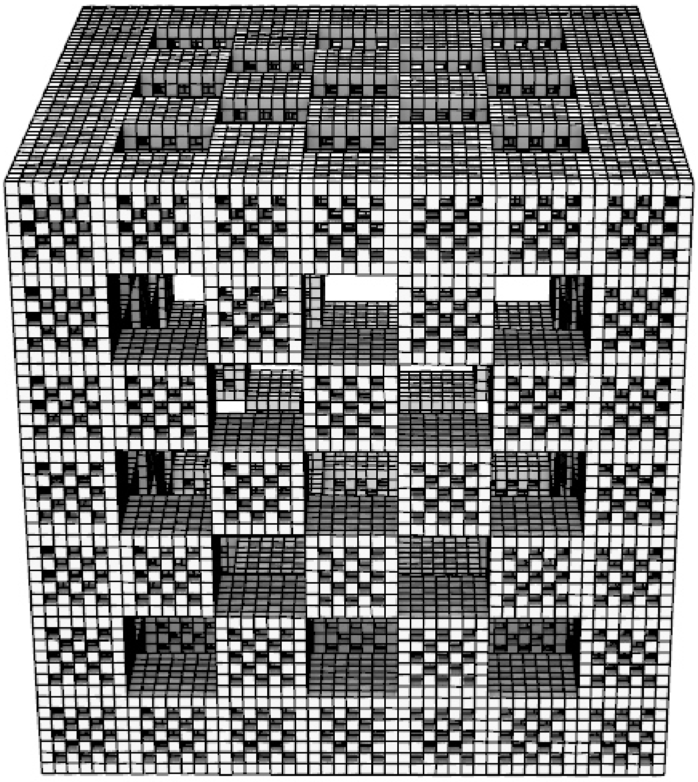

Figure 18: Chessboard

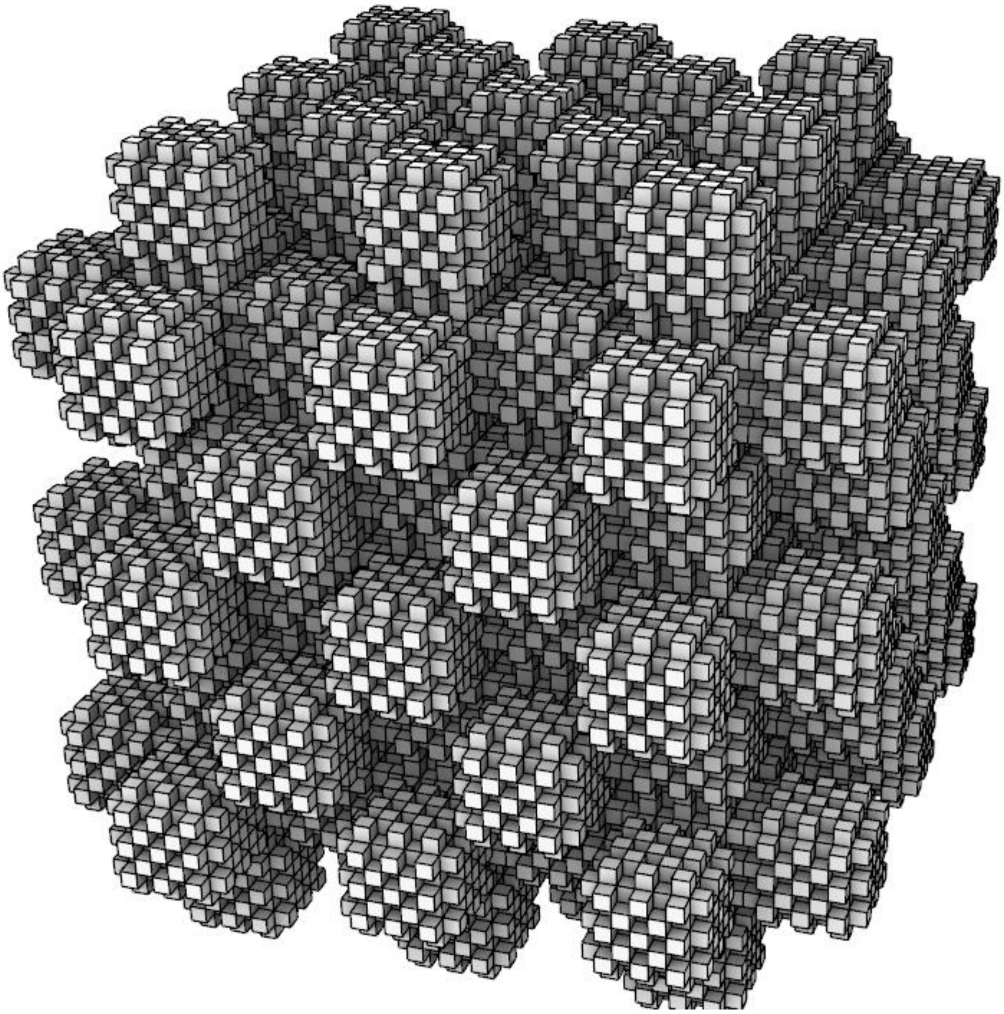

Figure 19: And his dual

When considering fractals with higher dimensions, it becomes more challenging for a computer to construct them beyond a few iterations, but the algorithm remains the same regardless of the size of the cube, whether it's 2x2x2 or 1000x1000x1000.

Regarding the dimensions of volume, in each iteration, if $t=n^3$ where n is the number of cubes of this side, t represents the total number of cubes. If s is the number of zeros in a particular case, we have that the fraction (t-s)/t is less than 1, and in each iteration, we multiply by a number less than 1. $V_n=((t-s)/t)^n$ in the limit is zero. The fractal has volume zero. His fractal dimension D =log(t-s)/log(3)

What can be said of a figure where each iteration uses different cubes of varying sizes? See Figure 11, for example. It would have a larger number of symmetrical shapes than the others, easily obtainable, making it an incredibly complex set of shapes.

Conclusion

Fractal geometry provides a powerful tool for understanding complex and self-similar shapes. In this paper, we have proposed a method for generating Menger-Diaz fractals, which is an expansion of Menger-type fractal constructions. We have shown how to apply this method to generate a variety of shapes and discussed the concept of a recursive "atom" that is fundamental to the Menger process. The proposed method can be used to create intriguing designs in various fields, including architecture, art, and computer graphics. Moreover, we have presented a detailed discussion of the algorithm for generating the Sierpinski tetrahedron using our proposed method, which can be extended to other polyhedra besides tetrahedra. Overall, our study contributes to the understanding of fractal geometry and opens up new avenues for exploring the possibilities of Menger-Diaz fractals.

Some ideas are in some preprints, books and articles. But no one seriously considers representation with a nested list of existence vectors or cube. And then our algorithm doesn't work. It is this trivial representation that is the core of our algorithm. If you don't understand that, you can't understand the algorithm. Thus, many approximations are forgotten and lost. In his honor we do not put a bibliography.

But, from now on, we propose to publish on the Internet all the artistic figures or those of scientific interest, produced with our algorithm, with the name of their authors. That will be our future bibliography.

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[1] Diaz, M. & Diaz, L. (2023). *An Algorithm for Extending Menger-Type Fractal Structures*. In preprint. https://easychair.org/publications/preprint/d9Sc

[2] Alsina, C., & Nelsen, R. B. (2015). *A Mathematical Space Odyssey: Solid Geometry in the 21st Century* (1st ed., Vol. 50). Mathematical Association of America. http://www.jstor.org/stable/10.4169/j.ctt15r3znz