Synthesis/Programming of Hopfield Associative Memory

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***Abstract*— In this research paper, the relationship between eigenvectors (with { +1, -1 } components) of synaptic weight matrix W and the stable/anti-stable states of Hopfield Associataive Memory is established. Also synthesis of W with desired stable/anti-stable states using spectral representation of W in even/odd dimension is discussed when the threshold vector is a non-zero vector. Freedom in choice of eigenvalues is capitalized to improve noise immunity of Hopfield Neural Network. Also, the problem of optimal synthesis of Hopfield Associative memory is presented.**

I. Introduction

Scientific thinking motivated researchers to arrive at mathematical models if natural systems. In one such effort, McCulloch-Pitts proposed a model of single neuronal cell. This model lacked training ability since the Synaptic weights (arising in artificial neuron model) are fixed at certain values. Rosenblatt improved McCulloch-Pitts model by allowing synaptic weights to vary during the training process. Such a model of neuron is called a Perceptron. He proposed and proved the convergence of Learning Law associated with perceptron (the so-called Perceptron Learning Law) when the patterns belonging to two classes are linearly separable. As a natural generalization, Single Layer Perceptron was proposed to classify patterns belonging to multiple classes when they are linearly separable. Minsky proposed XOR problem, which showed that XOR gate cannot be synthesized using a SLP (since the patterns are not linearly separable). Werbos proposed Multi-Layer Perceptron (using backpropagation algorithm) that enables classification of non-linearly separable patterns.

In an effort to model biological memory, Hopfield proposed a neural network which acts as an Associative Memory. Giles et.al proved a convergence theorem (by associating energy function with network dynamics) that confirms convergence in serial mode of operation. This theorem confirmed that the associated neural network acts as an associative memory. Hopfield naturally proposed the problem of synthesizing an artificial neural network with programmed stable states. This is the so called "Hopfield Neural Network Synthesis Problem". This research paper is an effort to solve such problem.

This research paper is organized as follows. In section 2

II. Review of Related Research Literature

Discrete time Hopfield Neural Network is a homogeneous (no external input) nonlinear dynamical system with the initial condition being a vector of {+1's, -1's} and the state space being symmetric, unit hypercube. The Artificial Neural Network (ANN) is represented by an undirected, weighted graph whose vertices are neuronal nodes and the weights are symmetric between the nodes. Each node is associated with a real valued threshold. Thus, the Ann is represented by a symmetric weight matrix, Wand threshold vector $\overline{T}$.

Let $V_i(n)$ represent {+1 or -1} valued state of 'i'th neuron and $\overline{V}(n)$ represent the state vector of the dynamical system (lying on symmetric, unit hypercube). Such an ANN operates in the following modes of operation.

$V_i(n+1) = \text{Sign}\{\sum_{j=1}^{N} W_{ij}\ V_j(n) - T_i\}$ Serial Mode

i.e. $\overline{V}(n+1) = \text{sign}\begin{cases} +1 & if\ Z \geq 0 \\ -1 & if\ Z < 0 \end{cases}$

In serial mode of operation, state of only on neuron is updated at any time instant, whereas in the fully parallel mode of operation, state of all the neurons is updated at any time instant. Thus, in the parallel mode of operation, we have

$\overline{V}(n+1) = \text{sign}\{\overline{W}\overline{V}(n) - \overline{T}\ \}$

All the other nodes of operation are called partial parallel nodes. In the above nodes of operations sign(.) is signum functions.

Definition: A state v(n) is called stable state if $\overline{V}(n) = \text{sign}\{\overline{W}\overline{V}(n) - \overline{T}\ \}$

Definition: Suppose $\overline{U}$ is a stable state. Then the stable state associated with $\overline{U}$ is $\overline{U}^T W \overline{U}$ i.e. quadratic from/energy value Similarly, anti-stable values are defined.

Now, we summarize the synthesis of W (and hence the Hopfield Associative Memory) using above two lemmas relating eigenvectors with stable/anti-stable states.

The following result is well known from linear algebra Every symmetric matrix has a spectral representation of the following form $W = PDP^T$, where 'P' is an orthogonal matrix whose columns are right eigenvectors that form an orthonormal basis.

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The spectral representation of W can be expressed in the following equivalent form

$W=\sum_{i=1}^{s} \lambda_i(\overline{f}_i)(\overline{f}_i)^T - \sum_{i=s+1}^{N} \mu_i(\overline{f}_i)(\overline{f}_i)^T$, where $\overline{f}_i$ are the right eigenvectors (column vectors) of W corresponding to eigenvalues{ λ_i, μ_i }.

It is thus clear that synthesis of W requires orthogonal corners of hypercube that constitute right eigenvectors of it. Such a problem was attempted by Sylvester, Hadamard who provided interesting solution. The detailed explanation of synthesis of such orthonormal basis requires the following definition.

Definition: Hadamard Matrix, H_m is of order 'm' satisfies the following equation 'H_m x $H^T{}_m$ = n I_m'.

Sylvester construction enables synthesis of Hadamard matrices of order n = 2^m for an integer 'm'. Thus, orthogonal stable states can be chosen as columns of Hadamard matrix. But dimension, of H_n must be a power of two.

Thus, the columns of normalized Hadamard matrix provide the desired orthonormal basis.

1) It readily follows (from Lemma 1) that any two orthonormal vectors of dimension N (lying on the hypercube) differ in exactly N/2 places.
2) Hadamard conjectured that Hadamard matrices of dimension 4k, exist for each integer K. But constructing Hadamard matrices in a given dimension is a difficult problem.
3) In dimension, $N=2^m$ for $m\geq 2$, Sylvester provided an interesting construction to arrive at Hadamard Matrices. In his construction H_2 is a Hadamard matrix of dimension 2.

Thus, once the dynamical system reaches a stable state, there will be no further change of state.

One of the important features of Hopfield neural networks is that if the diagonal elements of W are all non-negative, in the serial mode of operation, the neural networks converge to a stable state and in the fully parallel mode of operations, either the network converges or a cycle of length almost two is reached. This feature leads to the operation of Hopfield neural network as an associative memory(Hopfield Associative Memory).

Hopfield formulated the problem of synthesizing a HAM with certain stable states, say ($\overline{u}_1, \overline{u}_2, \ldots\ldots, \overline{u}_s$,).As an …..

In this programming/synthesis solution, the desired corners of hypercube are mutually orthogonal. The synaptic weight matrix is given by

$$W=\sum_{i=1}^{s}(\overline{u}_i\,\overline{u}_i^T - I)$$

Where I is an N x N Identity matrix

In this synthesis solution, the threshold vector $\overline{T}\equiv\overline{0}$ (i.e. a zero vector).Since $\overline{u}_i$'s are mutually orthogonal, We have

$$Wu_i = (N\text{-}S)\, u_i \text{ for } 1 \leq i \leq S$$

If N > S, it follows that Sign(W u_i)= $\overline{u}_i$ i.e. $\overline{u}_i$ is a stable state. This approach……

As an effort to solve the synthesis problem, Hopfield suggested an outer-product rule to arrive at the synaptic weight matrix with desired corners of hypercube as stable states. This approach was utilized by other researchers to synthesize the desired synaptic weight matrix. But it was shown by Bruck et.al that the outer product rule-based synthesis of synaptic weight matrix leads to exponentially many spurious stable states. After careful examination of logical basis of outer product rule, the author utilized spectral representation of synaptic weight matrix (using orthogonal corners of hypercube as eigenvectors) to arrive at synaptic weight matrix with desired stable states [1]. But the results in [1] assumed that the dimension of W is even and the threshold vector, $\overline{T}$ is zero vector.

III. Towards Optimal Synthesis of Hopfield Neural Networks in Even Dimensions

- **Eigenvectors of W with {+1, -1} Components: Stable/Anti-Stable States:**

In the synthesis of synaptic weight matrix using outer product rule, Hopfield utilized some orthogonal corners of hypercube as desired stable states. After understanding the essential idea behind Hopfield's Synthesis approach, the author utilized 'N' orthogonal corners of hypercube (with N being even) as the orthonormal basis of eigenvectors in the spectral representation of synaptic weight matrix, W. Detailed results of synthesis in even dimension are reported in [1]. The results assumed that the threshold vector, $\overline{T}\equiv\overline{0}$ i.e., a vector of zero elements.

We now generalize an essential lemma proved in [RaM]. Let d_H(X, Y): Hamming distance between{X,Y}, where X,Y are corners of hypercube.

***Note:* If dimension N is even, d_H(X, Y) is even, while if dimension N is odd, d_H(X,Y) is odd.(doubt)**

We briefly summarize the results reported in [1]. In [1], the main idea is based on the relationship between corners of hypercube that are eigenvectors of W corresponding to positive eigenvalues and the stable states. The following Lemma proved in **[RaM]** summarizes the relationship.

Lemma 2: Let $\overline{u}$ be a corner of hypercube that is an eigenvector of W corresponding to positive eigenvalue 'λ'. Then $\overline{u}$ is also a stable state when $\overline{T}\equiv\overline{0}$.

Proof: Refer **[RaM]**.

We prove a more general lemma in the following discussion.

In **[5]** motivated by above lemma the concept of ANTI-STABLE STATE is introduced.

Definition: $\overline{V}$ is an anti-stable state of Hopfield neural network with synaptic weight matrix, W if and only if

$$\overline{V} = -\text{Sign}(W\overline{V})$$

Lemma 3: Let $\overline{V}$ be a corner of hypercube that is an eigenvector of W corresponding to a negative eigenvalue "-μ". Then $\overline{V}$ is also an anti-stable state when $\overline{T}\equiv\overline{0}$.

Proof: Follows the same arguments as lemma 2.

- **Spectral Representation of W: Hadamard Matrices**:

Now, we summarize the synthesis of W (and hence the Hopfield Associative Memory) using above two lemmas relating eigenvectors with stable/anti-stable states.

The following result is well known from linear algebra
Every symmetric matrix has a spectral representation of the following form $W= PDP^T$, where ‘P’ is an orthogonal matrix whose columns are right eigenvectors that form an orthonormal basis.

The spectral representation of W can be expressed in the following equivalent form

$W=\sum_{i=1}^{s} \lambda_i (\overline{f_i})(\overline{f_i})^T - \sum_{i=s+1}^{N} \mu_i (\overline{f_i})(\overline{f_i})^T$, where $\overline{f_i}$ are the right eigenvectors (column vectors) of W corresponding to eigenvalues$\{ \lambda_i, \mu_i \}$.

It is thus clear that synthesis of W requires orthogonal corners of hypercube that constitute right eigenvectors of it. Such a problem was attempted by Sylvester, Hadamard who provided interesting solution. The detailed explanation of synthesis of such orthonormal basis requires the following definition.

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Thus, the columns of normalized Hadamard matrix provide the desired orthonormal basis.

4) It readily follows (from Lemma 1) that any two orthonormal vectors of dimension N (lying on the hypercube) differ in exactly N/2 places.
5) Hadamard conjectured that Hadamard matrices of dimension 4k, exist for each integer K. But constructing Hadamard matrices in a given dimension is a difficult problem.
6) In dimension, $N=2^m$ for $m \geq 2$, Sylvester provided an interesting construction to arrive at Hadamard Matrices. In his construction H_2 is a Hadamard matrix of dimension 2.

$$H_{2^{m+1}} = \begin{bmatrix} H_{2^m} & H_{2^m} \\ H_{2^m} & H_{2^m} \end{bmatrix} \quad \text{for } m \geq 1.$$

Thus, using such construction, we can synthesize Hadamard matrices of dimension $N=2^L$ for any integer L ≥ 1.

Thus, in view of spectral representation of W, we can synthesize Hopfield Associative memory using columns of Hadamard matrices in dimension $N=2^L$ for any integer L ≥ 1.

7) From Lemma 1, it readily follows that Hadamard matrices exist only if the dimension, N is an even number and N=4K, for integer K.
8) Synthesizing Hopfield Neural Network in other dimensions ($N \neq 2^m$ and N=4k for integer k) is a difficult problem. Recent results on Hadamard matrices (e.g. Payley construction etc.) provide construction procedure in some other dimensions. No general method of constructing H_{4k}, for every integer k is currently known.

- **Non-Zero Threshold Vector: Programming Problem Solution:**

Now, we consider the HAM synthesis problem in even dimension (i.e. N is even) when the threshold vector, $\overline{T}$ is a Non-Zero vector. As we can expect, the synthesis requires constraints on the positive and negative eigenvalues.

Lemma 3: Let the threshold vector of Hopfield Associative Memory(HAM), $\overline{T} \neq \overline{0}$. Let $\overline{u}$ be an eigen vector of $\overline{W}$ corresponding to positive eigenvalue, λ which is also a corner of unit hypercube.

Then $\overline{u}$ is a stable state of HAM if $|\lambda| > \max_i |\mathbf{T_i}|$ i.e. $|\lambda| > |T_{max}|$, i.e., Maximum of absolute values of components of $\overline{T}$.

Proof: Sign$\{\overline{W}\,\overline{u} - \overline{T}\}$ = Sign$\{\lambda \overline{u} - \overline{T}\}$
For $\overline{u}$ to be a stable state, we require that Sign$\{\lambda \overline{u} - \overline{T}\} = \overline{u}$. We keep in mind that components of vector $\overline{u}$ are +1 or -1. We thus consider Sign$\{\lambda u_i - T_i\}$. We effectively have two cases: u_i = +1 or -1.

- $u_i = +1$ case: Since $\lambda > 0$, if $T_i < 0$ then, always Sign$(\lambda u_i - T_i) = u_i$.
- $u_i = +1$ case: Since $\lambda > 0$, if $T_i > 0$ then, for Sign$(\lambda u_i - T_i) = u_i$, we require that $\lambda > |T_i|$ for all i. Thus, for two such cases $\lambda >$ Maximum positive component of $\overline{T}$.
- $u_i = -1$ case: Since $\lambda > 0$, if $T_i < 0$ then, for Sign$(\lambda u_i - T_i) = u_i$, we require that $-\lambda <$ Smallest negative component of $\overline{T}$.
- $u_i = -1$ case: Since $\lambda > 0$, if $T_i > 0$ then always Sign$(\lambda u_i - T_i) = u_i$. Thus, for two such cases $|\lambda| > \max_i |T_i|$ or equivalently $|\lambda| > |T_{max}|$ i.e., Maximum of absolute values of components of $\overline{T}$ Q.E.D .
- Now, we derive constraints on negative eigenvalues in the following lemma.

Lemma 5: Let the threshold vector of Hopfield Neural Network, $\overline{T} \neq \overline{0}$. Let $\overline{V}$ be an eigenvector of $\overline{W}$ corresponding to negative eigenvalue, -μ which is also a corner of unit hypercube. Then $\overline{V}$ is an anti-stable state of HAM if $|\mu| > \max_i |T_i|$ i.e., $|\mu| > \max_i \{|T_i|\}$ i.e. Maximum of absolute values of components of $\overline{T}$.

Proof: Sign$\{W\overline{V} - \overline{T}\}$ = Sign$\{-\mu\overline{V} - \overline{T}\}$. For $\overline{V}$ to be an anti-stable state, we require that Sign$\{W\overline{V} - \overline{T}\} = -\overline{V}$. We realize that the components of $\overline{V}$ are {+1 or -1}. We thus consider Sign$\{-\mu\, v_i - T_i\}$ (for all i). We effectively have two cases: v_i = +1 or -1.

- v_i= +1 case: Since $-\mu < 0$, if $T_i > 0$, then always Sign$\{-\mu v_i - T_i\} = -\mu_i$.
- v_i = +1 case: Since $-\mu < 0$, if $T_i < 0$, then, for Sign$\{-\mu v_i - T_i\} = -\mu_i$, we require that $-\mu <$ smallest negative component of $\overline{T}$ .Thus, for two such cases, we require that $|\mu| > \max_i |T_i|$.

Now, we consider the other case:

- v_i= -1 case: Since -μ<0, if $T_i > 0$, then, for Sign{-μv_i-T_i} = -v_i, we require that μ > Maximum positive component of $\overline{T}$.
- v_i= -1 case: Since -μ < 0, if $T_i < 0$, then always Sign{-μv_i-T_i} =-v_i. Thus, for two such cases |μ| > $\max_i |\mathbf{T_i}|$ Q.E.D .

- **Solution to the Synthesis Problem: Choice of Eigenvalues:**

Using lemmas 3 and 4 with positive/negative eigenvalues suitably chosen (to satisfy the specified constrains), we can synthesize, W in the following manner

$$W=\sum_{i=1}^{s} \lambda_i \overline{u_i} - \sum_{i=s+1}^{N} \mu_i \overline{V_i}$$

In such synthesis approach (N is even), the synaptic weight matrix will be a diagonally dominant matrix with all the diagonal elements being equal to the sum of eigenvalues (i.e. Trace(W)). Due to diagonal dominance of W, in this case in this case, all the corners of hypercube will be stable states (as can be readily seen). Thus with 'N' desired/ programmed stable states, there will be '2^N-N' "Spurious" stable states (exponentially many). Hence, in such approach, the minimum distance between stable states is ONE and hence the synthesis approach is a very poor one. Such synthesis procedure introduces large number of spurious stable states(as every corner of hypercube will be a stable state since W will be diagonally dominant matrix).

Now, we take a closer look at the problems in the above Synthesis procedure and improve it to achieve better NOISY IMMUNITY properties of resulting associative memory.

MAIN IDEA: To capitalize the freedom in choice of eigenvalues of W.

Step 1: In even dimension (i.e. N is even), using Sylvester construction, arrive at orthonormal basis of corners of hypercube (possibly only if $N=2^m$ for integer 'm').

Step 2: To avoid the problems in previous Synthesis procedure, ensure that the Trace(W)=0 i.e. Sum of eigenvalues is zero. For instance, positive, negative eigen values can be located symmetrically around zero. In section V, we specify a procedure to choose the eigenvalues. As discussed earlier, the smallest positive eigenvalue, the smallest negative eigenvalue must specify the conditions in lemma 3,4,5.

IV. SYNTHESIS OF HOPFIELD NEURAL NETWORK IN ODD DIMENSION

- **Odd Dimension: No Orthogonal Corners of Hypercube:**

In [RaM], it was stated that Hopfield Associative Memory Synthesis is NOT possible when N is odd (in fact HAM is claimed NOT to exist in odd dimension). This conclusion requires clarification. From lemma1, it readily follows that no two corners of hypercube are ORTHOGONAL when the dimension (of vectors), N is odd. Since the desired corners of hypercube that are stable/anti-stable states must be orthogonal (in the spectral representation of W), at most ONE CORNER of hypercube can be eigenvector corresponding to positive/negative eigenvalue. Such a desired corner will be stable/anti-stable state of synthesized HNN in odd dimension. In the definition of Hopfield and other researchers, all other stable/anti-stable states of synthesis W are "Spurious". This restriction in odd dimension may be an advantage in the following sense.

A. *Hopfield Associative Memory with ONE DESIRED STABLE STATE:*

In some applications of associative memories with EXACTLY ONE DESIRED MEMORY it is required to synthesize associative memory with i.e. in such applications, Hopfield Associative Memory can be synthesized in odd dimension (as discussed above). Such a desired memory (stable state/anti-stable state) can be chosen to correspond to any positive/negative eigenvalue.

- **$\bar{e}$ as Programmed Stable State:**

Also, the desired stable state can be the specific state $\bar{e}$= $[11\ldots..1]^T$ i.e. Column vector all of whose components are 1, corresponding to any positive eigen value, with $\overline{T} \equiv \bar{0}$.

Since the set of eigenvectors of W (even when N is odd) must form an orthonormal basis; if $\overline{f}$ is an eigenvector of W different from $\bar{e}$, $\overline{f}^T\bar{e}$ = 0 i.e. Sum of positive components of $\overline{f}$ must equal sum of negative components of $\overline{f}$. This in turn implies that L^1- norm ($\overline{f}$) must be divisible by 2.

- **None of the Eigenvectors are Stable States:**

Suppose none of the corners of hypercube is a "desired" stable state (programmed as an eigenvector in the spectral representation of W). Then, the orthonormal basis of eigenvector i.e. columns of orthogonal matrix, P (all of which don't lie on hypercube) leads to the spectral representation of W i.e.,

$$W = P\, D\, P^T = \sum_{i=1}^{S} \lambda_i\, \bar{g}_i\, \bar{g}_i^T - \sum_{i=S+1}^{N} \mu_i\, g_i\, \bar{g}_i^T$$

Thus, in this case, all the resulting stable states are "spurious stable states". Also, from Lemma 1, it follows that in this case $X^TY \neq N/2$ for any spurious stable states X, Y (as they are not orthogonal i.e. $X^TY \neq N/2$).

V. HOPFIELD NEURAL NETWORKS : NOISE IMMUNITY PROPERTIES

- **Signed Eigenvectors: "Maximal Stable States:**

We first consider the case, where N is odd. We need the following definition based on the fact that in the serial mode HNN converges to a stable state starting in any initial condition.

Definition: The domain of attraction of a stable state is the set of all initial corners of hypercube that converges to it in serial or parallel mode of operation.

The following lemma follows from Theorem () in [5]. We assume positive definite W for convenience.

Lemma 6: Let the orthonormal basis of eigenvectors of weight matrix, W be {$\bar{g}_1$, $\bar{g}_2$, …., $\bar{g}_N$}.

Let $\bar{f}_i$ = Sign($\bar{g}_i$) for 1≤ i ≤ N. Also, let {$\bar{h}_i$: 1≤ i ≤ N} be the stable states reached with initial states of HNN being {$\bar{f}_i$: 1≤ i ≤ N} respectively. Then, $\bar{h}_i$'s are the stable states of W with the associated stable values being the 'N' largest ones among all possible stable values. We call them 'N' maximal stable states.

The above lemma and theorem () in **[5]** are based on the following well known Rayleigh's theorem.

Rayleigh's theorem: The local optima of the quadratic form associated with a symmetric matrix A on the unit Euclidean hypersphere (i.e. X ∶ X^TX = 1) occur at the eigenvectors with the corresponding value of the quadratic form being the eigenvalues.

- **"Desired Stable States" in Odd Dimension:**

A. Possible Convention/Notation:

In the case of Hopfield Associative Memory with 'odd' number of neurons, {$\bar{h}_i$ ∶ 1≤ i ≤ N} can be considered as DESIRED STABLE STATES.

Note: If 'W' is indefinite (i.e. has positive as well as negative eigenvalues), then similar Lemma as the above one can be proved. Details are provided for brevity.

Note: With the above notation, determining Desired stable states when is odd involves the following steps:

i) Computing eigenvectors i.e. {$\bar{g}_i$ ∶ 1≤ i ≤ N},
ii) Computing f_i = Sign {$\bar{g}_i$ } for 1≤ i ≤ N, and
iii) Running HNN in serial mode with $\bar{f}_i$'s as initial states and arriving at $\bar{h}_i$'s.

Note: Unlike the case when N is even, with N being odd we are unable to predict the Hamming distance between the desired stable states.

- **Even Dimension: Energy Landscape of HAM:**

Now, we consider the case where N is even. From lemma 1 and synthesis procedure discussed in Section 3, all the programmed stable states i.e. eigenvectors of W that are corners of hypercube are at a Hamming distance N/2(i.e. any two desired stable states are at a Hamming distance exactly N/2 as in the case of SIMPLEX code).

Also, using Rayleigh's theorem, the 'N' stable values corresponding to N desired stable states are the local optimum values on the unit hypercube, unit Euclidean hypersphere.

Now, in the case where 'N' is even, the energy landscape has a beautiful interpretation. The following lemma uncovers the hidden pattern behind the energy landscape when $\bar{T} \equiv \bar{0}$(zero vector).

We explore the energy landscape to improve the error correction ability of the Hopfield Associative Memory (HAM).

Consider the HNN in which the threshold vector $\bar{T} \equiv \bar{0}$(zero vector). Hence the energy function associated with network dynamics becomes

$$E(n) = \bar{V}^T(n)\, W\, \bar{V}(n) \text{ (i.e. quadratic form)} \qquad ()$$

Consider any programmed/desired stable state (with synthesis of W as discussed above) as one among the 'N' eigenvectors of W.

Consider all {+1, -1} vectors which are at the same Hamming distance from any eigenvector/desired stable state. Let $\bar{U}$ be one such vector.

Lemma 7: The value of Energy associated with all such vectors $\bar{U}$(i.e. $\bar{U}^T$ W $\bar{U}$(a quadratic form)) all such points is a constant.

Proof: The main fact required in the proof is that all {+1, -1} vectors at a constant Hamming distance from a programmed/desired stable state are arrived at by permuting the components of other. In other words, suppose {$\bar{U}, \bar{V}$} are such vectors (at a constant Hamming distance from a stable state).

$\bar{V} = \bar{Q}\bar{U}$, where $\bar{Q}$ is a symmetric permutation matrix with Q^T= Q = I.

Let $\bar{P}$ be the orthogonal matrix (whose columns are desired/programmed stable states that are also eigenvectors).

Thus, the spectral representation of symmetric synaptic weight matrix W becomes W = $\bar{P}\bar{D}\bar{P}^T$, with $\bar{D}$ being the diagonal matrix of real eigenvectors of W.

$$V^T \bar{W}\, \bar{V} = \bar{V}^T P\, D\, P^T \bar{V}$$
$$= \bar{U}^T Q^T P\, D\, P^T Q\, \bar{U}$$
$$= \bar{U}^T Q^T W\, \bar{Q}\, \bar{U}, \text{ since } \bar{Q} \text{ is a symmetric matrix.}$$

Fact: Q: Symmetric, permutation matrix implies that QB = BQ, for any matrix B or Q^TB = B Q^T

$$V^T \bar{W}\, \bar{V} = \bar{U}^T Q^T W\, Q\, \bar{U}$$
$$= \bar{U}^T W\, Q^T Q\, \bar{U}$$
$$= \bar{U}^T W\, \bar{U} \qquad \text{Q.E.D.}$$

- **Capitalizing Freedom in Choice of Eigenvalues: Basins of Attraction**:

Now, we choose the eigenvalues (capitalizing freedom in their choice) such that all the corners of hypercube which are at a distance of at most d_{min} from the associated stable states are in the domain of attraction of the associated stable states we use the fact that from the proof of convergence theorem, in the serial mode of operation, the energy function is non-decreasing and reaches a local/global maximum once the stable state is reached. Using above lemma, if we ensure that the energy values corresponding to different initial condition vectors(in different domains of attraction) are sufficiently distinct, then the desired energy landscape is synthesized.

Let {$\bar{U}_1$, $\bar{U}_2$, …., $\bar{U}_N$} be any one of the many vectors at a Hamming distance[d_{min} =$\left\lfloor\frac{N-2}{4}\right\rfloor$] from the corresponding programmed/ desired stable states {$\bar{V}_1$, $\bar{V}_2$,….., $\bar{V}_N$}, which are also eigenvectors of W.

Let W = P D P^T be the spectral representation symmetric synaptic weight matrix W. Let $\bar{P}$ be the orthogonal matrix whose columns are {$\bar{V}_1$, $\bar{V}_2$, …., $\bar{V}_N$}. The columns of $\bar{P}$ form a (normalized) orthonormal basis for 'N-d' Euclidean Space, $\mathbb{R}^N$. Also, elements of diagonal matrix D are the eigenvalues.

$$\hat{U}_1 = \frac{1}{\sqrt{N}}\bar{U}_1 = \bar{P}\bar{C}_1 \quad (\hat{U}_1^T \bar{U}_1 = \bar{C}_1^T C_1 = 1) \qquad \textbf{()}$$

i.e. $\hat{U}_1$ lies on N-dimensional Euclidean unit hypersphere.

$$\hat{U}_1^T W\, \hat{U}_1 = C_1^T P^T P\, D\, P^T P\, C_1$$
$$= \sum_{i=1}^{N} c_i^2 \lambda_i, \text{ with } \sum_{i=1}^{N} c_i^2 = 1 \qquad (10.5)$$

Now, we provide an approach to capitalize the freedom in choice of eigenvalues.

From above discussion, we know {U_1, …, U_N}. Hence, we know {$\hat{U}_1$, $\hat{U}_2$, ….., $\hat{U}_N$} as well as the orthogonal matrix $\bar{P}$.

We know $\bar{c}_i$ = P^T $\hat{U}_i$ = P $\hat{U}_i$ (since the Sylvester construction based Hadamard matrix P is symmetric).

Let $\hat{U}_i^T$ W $\hat{U}_i$ = S_i = $\sum_{j=1}^{N} c_{ij}^2 \lambda_j$.

We have the following system of linear equations,

$$\begin{bmatrix} c_{11}^2 & c_{12}^2 & \cdots & c_{1N}^2 \\ c_{21}^2 & c_{22}^2 & \cdots & c_{2N}^2 \\ \vdots & \vdots & \vdots & \vdots \\ c_{N1}^2 & c_{N2}^2 & \cdots & c_{NN}^2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_N \end{bmatrix}$$

Equivalently, $\bar{C}\,\bar{\lambda} = \bar{S}$

$\bar{C}$ is a stochastic matrix, since $\sum_{j=1}^{N} c_{ij}^2 = 1$, for all 'i'.

The elements of vector $\bar{S}$ are chosen to be interesting sequence of increasing numbers so that the domains of attraction corresponding to coding spheres of stable states, eigenvectors.(10.6)

- **Singular W: Noise Immunity of HAM**:

In the synthesis approach discussed above (in the case where N is even or N is odd), it is assumed that the matrix W is NON-SINGULAR (i.e. none of the eigenvalues is zero).

In practical applications, it may be needed to program strictly less than N stable/anti-stable states when W is an NxN matrix.Let us now consider the case where W is singular. Let the dimension of null space of W be 'K' (i.e. there are 'K' linearly independent vectors in the null space of W). Let $\bar{f}$ be a vector in the null space.

$$\overline{W}\ \bar{f} \equiv \bar{0} \Rightarrow W(-\bar{f}) = -W\,\bar{f} \equiv \bar{0}$$

i.e. $-\bar{f}$ is also in the null space of W. By convention, let Sign($\bar{0}$)= $\bar{e}$, where $\bar{e}$ is a column vector all of whose components is '1'.

Further, suppose 'W' is a non-negative matrix(W$\not\equiv \bar{0}$). Then we have that Sign(W$\bar{e}$) = $\bar{e}$ i.e. $\bar{e}$ is a stable state.

Hence in this case (i.e. W is a singular non-negative matrix), all the corners of hypercube which are in the null space of W are in the domain of attraction of stable state '$\bar{e}$'.

Suppose W is singular, but not a non-negative matrix. Further, let $\bar{e}$ be in the domain of attraction of stable state $\bar{h}$. Hence, in this case, all the vectors in the null space of $\overline{W}$ are in the domain of attraction of $\bar{h}$ (stable state).

Note: Similar results can be derived for the case of anti-stable states. Details are avoided for brevity.

Note: In the case where 'N' is even, the Hopfield Associative Memory (HAM) can be synthesized with, say "N-S" stable states (S>1) and the associated 'W' is singular.

The above results apply in this case.

We are naturally led to singular 'W' in the following case also. Let $2^m < N < 2^{m+1}$ for an integer 'm'.

To synthesize 'W', we use N columns of Hadamard matrix H_2^{m+1} with suitable 'N' eigenvalues and remaining eigenvalues (2^{m+1}-N) to be zero.

Thus, such a synthesis approach ensures "good" noise immunity properties of associated HAM.

*Note***: Conversion of Synthesis of W in odd dimension to even dimension.**

Suppose 'N' is an odd integer. Then, consider the closest integer, M such that M= 2^l for some integer with M>N. Then synthesize 'MxM' singular synaptic weight matrix with 'M-N' zero eigenvalues and the other 'N' eigenvalues being suitably chosen with the corresponding eigenvectors being columns of Hadamard matrix, $H_2^l = H_M$. In this case, the null space of W is spanned by the remaining 'M-N' corners of hypercube that are orthogonal.

For the sake of completeness, we briefly summarize the results of Bruck et.al, relating Hopfield Neural Network with the associated Graph-theoretic code.

Bruck et.al showed that the stable states of HNN are naturally related to the codewords of graph theoretic code (which correspond to the cuts) associated with graph defining HAM. Every codeword is associated with a bi-partite graph where the stable state is obtained by placing '+1' for vertices on one side of cut and '-1' for vertices on the other side of graph.

VI. Formulation of Optimal Synthesis(of W) Problem

Let 'K' be the number of desired stable states (K could be strictly greater than N).

Synthesize W with K corners of hypercube as stable states that are at maximum possible minimum Hamming distance (d_{min}).

In such optimal synthesis problem, the following two problems naturally arise.

1. Given an integer 'K', what is the maximum possible value of 'd_{min}'?

We conjecture that if K≥N, then maximum possible value of d_{min} = N/2.

2. Given a required "d_{min}" value, what is the maximum possible value of K?

VII. CONCLUSION:

In this research paper, the synthesis of Hopfield Associative Memory with desired stable/anti-stable states is discussed (using spectra representation of W with {+1, -1 } component eigenvectors). The optimal synthesis of Hopfield Neural Network is discussed. Capitalizing the freedom in choice of eigenvalues, the noise immunity of HAM is improved.

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