Hopfield Neural Network and Anisotropic Ising Model

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Abstract. The probabilistic Hopfield model known also as the Boltzman machine is a basic example in the zoo of artificial neural networks. Initially it was designed as a model of associative memory, but played a fundamental role in understanding the statistical nature of the realm of neural networks. The close relation between the Boltzman machine and the Ising model was a challenging observation in [1]. In this note we go further, we establish another type of structural similarity between these models sharing the methods of the Bethe ansatz family of integrable statistical mechanics. We examine the asymmetric model on the triangular lattice with arbitrary weights. We show that the probability of passing a trajectory in time dynamics obeys the Gibbs distribution with a partition function of the Ising model on the cubic lattice with additional weights on diagonals.

1 Introduction

The field of artificial neural networks was born in the 40s, underwent intensive development in the 60s, was actively fueled by the ideas of statistical physics in the 80's, and in applications made a giant leap in the last 10 years. This leap is largely due to the intensive growth in the performance of modern parallel computing systems.

The field was born as an attempt to systematize the mechanisms of nervous activity of living beings. It turned out that learning rules like the Hebb rules are so universal that artificial neural networks, which differ significantly from natural ones, began to show high results in generalizing ability.

The actual formulation of the problem of evaluating the generalizing ability of the network arose due to the introduction of ideas of statistical mechanics in the field of neural networks.

The Hopfield network has its proper application as one of the fundamental models of content-addressable associative memory system [1]. This model have demonstrated some very interesting collective properties of the neural network behavior and is used in artificial neural networks as like as in neurophysiology [3] in research of memory capacities, memory

retrieval process, short-term plasticity, working memory properties and many other questions.
The main goal of this work is to transfer methods for the exact solution of statistical physics models to neural network models. In this regard, it is necessary to recall what results are considered important in this area. The main model example is the two-dimensional Ising model, which has been solved explicitly by many groups of scientists using different methods [4]. It has a critical behavior, that is, in the vicinity of the critical temperature, it has the property of a phase transition of the second kind. The memory capacity of the Hopfield model demonstrated a similar property of the phase transition in[5].
The most common method to solve such models is the Bethe ansatz in termodynamic limit. The principal goal of this paper is to establish a relation between the anisotroppic Hopfield model on the triangular lattice and the Ising on the appropriately chosen lattice. This work extends some ideas of [7, 8] to the anisotropic case. Another purpose of this note is the extension of the ideas of the cluster algebraic structures [12, 13] to the world of neural networks which could produce interesting invariants for the neural networks with respect to the mutaions of the "star-triangle" type.

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1.1 Hopfield model

The Hopfield model is represented by the complete graph with N vertexes (neurons) with a connectivity matrix W_{ij} characterizing the conductivity of the synapse between i-th and j-th neurons. At each time the system is characterized by its neurons states $\{x_i\}, i = 1, \ldots, N$ $x_i = \pm 1$. Our interest is focused on the network which undergoes the synchronous evolution in discrete time. In the deterministic version the state of the i-th neuron at the next step x'_i is determined by the formula:

$$x'_i = \begin{cases} 1 & \text{if } \sum_j W_{ij} x_j > t_i \\ -1 & \text{if } \sum_j W_{ij} x_j < t_i \\ x_i & \text{else.} \end{cases} \tag{1}$$

Here t_i is the threshold level for the activation of the i-th neuron. The probabilistic Hopfield model differs from the deterministic one by the property that the transition is performed with the probability specified by the Fermi sigmoid function

$$P(x', x) = \prod_i (1 + e^{-\beta x'_i (\sum_j W_{ij} x_j - t_i)})^{-1}.$$

For the threshold level $t_i = 0$ this expression can be rewritten as

$$P(x', x) = e^{-\frac{\beta}{2}\sum_{ij} W_{ij} x'_i x_j} / \sum_{x''} e^{-\frac{\beta}{2}\sum_{ij} W_{ij} x''_i x_j}. \qquad (2)$$

The similarity of such an expression with the partition function of the Ising model was remarked in [1].
We need to say some words about the learning stage in the Hopfield model. The Hebbian paradigm for the learning process of the Hopfield network on the set of m patterns $\{\epsilon^1, \ldots, \epsilon^m\}$ each of which is a vector

$$\epsilon^k = (\epsilon_1^k, \ldots, \epsilon_n^k)$$

is achieved instantaneously by fixing the weight matrix in a following way

$$W_{ij} = \frac{1}{n}\sum_{k=1}^{m} \epsilon_i^k \epsilon_j^k.$$

The work of the network in the recall stage consists of iterative time dynamics defined by the transformation probability (2).
The principal interplay of the Hopfield network and the Ising model is due to the energy functional. It turns out that for the symmetric weight function in asynchronous regime the expression

$$E = -\frac{1}{2}\sum_{i,j} W_{ij} x_i x_j - \sum_i t_i x_i$$

plays the role of the Lyapunov function, due to the time dynamics it either lower or stays the same. This observation allows to analyze the asymptotic behavior of the Hopfield model. The stable points for its time dynamics are hence the states of local minimum energy for the Ising model on the same lattice. In this paper we explore another relationship between two such models: the Hopfield network on a two-dimensional lattice and the Ising model on the 3-dimensional lattice, making one of the spacial directions relevant to the time evolution.

1.2 Time evolution and the Ising model

Let us consider the Hopfield model on the triangular lattice (Fig. 1) colored by 3 colors in $\mathbb{Z}/3\mathbb{Z}$. The nontrivial weights are defined by rule: the neuron with the color c is influenced only by the neurons with the colors $(c-1)\ mod\ 3$. This network is not symmetric despite the canonical definition.
This lattice can be viewed as a projection of the cubic lattice to the plane $i + j + k = 0$. The verteces with different colors represent the planes $i + j + k = c\ mod\ 3$. The temporal behavior of this lattice is equivalent to the 3 independent 3-dimensional cubic lattices. We then consider the one of these 3 lattices.

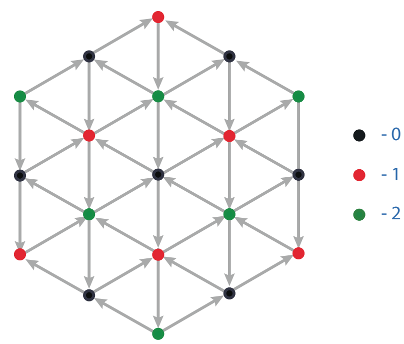

Fig. 1. Triangular lattice

The conditional probability that the model passes throw the states with free initialization data is:

$$P = \prod_{i+j+k=a}^{b} \left(1 + \exp(x_{ijk}(w^1_{ijk}x_{i-1jk} + w^2_{ijk}x_{ij-1k} + w^3_{ijk}x_{ijk-1})\right)^{-1}$$
$$= \prod_{i+j+k=a}^{b} \frac{\exp((x_{ijk}(w^1_{ijk}x_{i-1jk} + w^2_{ijk}x_{ij-1k} + w^3_{ijk}x_{ijk-1})/2)}{2\cosh((x_{ijk}(w^1_{ijk}x_{i-1jk} + w^2_{ijk}x_{ij-1k} + w^3_{ijk}x_{ijk-1})/2)}$$

Let us define a matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

which is a square root of 1

$$A^2 = 1.$$

Let us consider a transformation $f : w \to \widetilde{w}$

$$f(w^1, w^2, w^3) = (w^0, w^{12}, w^{13}, w^{23})$$

given by the system of equations

$$\begin{pmatrix} w^0 \\ w^{12} \\ w^{13} \\ w^{23} \end{pmatrix} = A \begin{pmatrix} \log\cosh((w^1 + w^2 + w^3)/2) \\ \log\cosh((w^1 + w^2 - w^3)/2) \\ \log\cosh((w^1 - w^2 + w^3)/2) \\ \log\cosh((w^1 - w^2 - w^3)/2) \end{pmatrix}. \tag{3}$$

Lemma 1 *Let* $f(w^1, w^2, w^3) = (w^0, w^{12}, w^{13}, w^{23})$ *than the following equation holds*

$$(\cosh((w^1 s_1 + w^2 s_2 + w^3 s_3)/2))^{-1} = \exp\left((w^0 + w^{12} s_1 s_2 + w^{13} s_1 s_3 + w^{23} s_2 s_3)/2\right) \quad (4)$$

$\forall s_i = \pm 1.$

Proof
Both sides of 4 are invariant with respect to the total change of signs $s_i \mapsto -s_i$. Hence it is sufficient to prove this statement for 4 combinations of spins with $s_1 = 1$. In this way we get a system of linear equation with the defining matrix A. The fact that it is square root of unity gives the result.
□

Remark 1 *The transformation* f *restricted to the last 3 variables* $F : (w^1, w^2, w^3) \mapsto (w^{12}, w^{13}, w^{23})$ *was known in the theory of the Ising model as a "star-triangle" transformation [9]. It appears to be a solution for the Zamolodchikov tetrahedron equation [10].*

Theorem 1 *The conditional probability 3 coincides with the Ising-type partition function:*

$$P = \prod_{i+j+k=a}^{b} \exp\left((x_{ijk}(w^1_{ijk} x_{i-1jk} + w^2_{ijk} x_{ij-1k} + w^3_{ijk} x_{ijk-1})/2\right) \times$$
$$\times \prod_{i+j+k=a}^{b} \exp\left((w^{12}_{ijk} x_{i-1jk} x_{ij-1k} + w^{13}_{ijk} x_{i-1jk} x_{ijk-1} + w^{23}_{ijk} x_{ij-1k} x_{ijk-1})/2\right)$$

where $(w^{12}_{ijk}, w^{13}_{ijk}, w^{23}_{ijk}) = F(w^1_{ijk}, w^2_{ijk}, w^3_{ijk})$.

Remark 2 *This model can be interpreted as an Ising model on the regular cubic lattice with additional diagonal edges with weights defined by* $(w^{12}_{ijk}, w^{13}_{ijk}, w^{23}_{ijk})$. *We illustrate the weight distribution on Fig. 2.*

1.3 Conclusion

The principal aim of this work concerns the possibility of the Bethe ansatz method application to the description of the critical behavior of the Hopfield neural network. This could be fruitful in such a technique as the simulated annealing in neural networks [11]. In this note we generalized a relation between the Hopfield model and the Ising one in the case of the completely anysotropic models. This is interesting for us also for the reason that this case allows to apply the methods of cluster algebraic structures [12] and [13] for the case of neural network models.

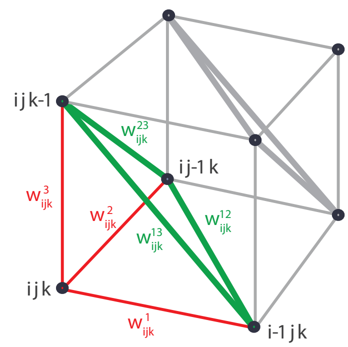

Fig. 2. Cubic lattice

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