Implicativity Versus Filtrality, Disjunctivity and Equality Determinants

Alexej Pynko

June 28, 2023

IMPLICATIVITY VERSUS FILTRALITY, DISJUNCTIVITY AND EQUALITY DETERMINANTS

ALEXEJ P. PYNKO

Abstract. Extending the notion of an *implicative system for* a class of algebras by admitting existential parameters, we come to that of [*restricted*, viz., parameter-less] one, {quasi-}varieties with {relatively} subdirectly-irreducibles {[i.e., those generated by subclasses]} with [restricted] implicative system being called *[restricted] implicative*. Likewise, a {quasi-}variety is said to be *{relatively} [sub]directly filtral/congruence-distributive*, if {relative} congruences /lattice of any [sub]direct product of its {relatively} subdirectly-irreducibles are/is filtral/distributive, pre-varieties (viz., abstract hereditary multiplicative classes) generated by subclasses with ⟨finite⟩ disjunctive system being called *⟨finitely⟩ disjunctive*. The main general results of the work are that any /{quasi-}equational {pre-}variety is /⟨finitely⟩ disjunctive iff it is {relatively} congruence-distributive with {its members isomorphic to subdirect products of relatively finitely-subdirectly-irreducible ones}/ and the class of its {relatively} finitely-subdirectly-irreducible members being "a universal /⟨first-order⟩ model class"|"hereditary /⟨and closed under ultra-products⟩", while any {quasi-}variety is [restricted] implicative it is {relatively} [sub]directly filtral iff it is {relatively} [⟨finitely-⟩]semi-simple (i.e., its {relatively} [⟨finitely-⟩]subdirectly-irreducibles are {relatively} simple) and [sub]directly congruence-distributive with the class of {relatively} simple members being "a [universal] first-order model one"|"[hereditary and] closed under ultra-products" [iff it is disjunctive and {relatively} finitely-semi-simple] if[f] it is {relatively} semi-simple and has [R]EDP{R}C. In particular, any finitely-generated /semi-simple variety of lattice expansions with hereditary class of subdirectly-irreducibles is disjunctive/"restricted implicative" that provides an immediate insight into "disjunctivity but not"/restricted implicativity /"and REDPC" for the finitely-generated "but not"/ semi-simple variety of "Stone algebras"/"distributive|De |Morgan lattices|algebras||lattices". Finally, we exemplify our general elaboration by applying it to the disjunctive non-implicative [quasi-]equational join (viz., the [quasi-]variety generated by the union) of the varieties of Stone algebras and De Morgan algebras/lattices as well as finding the lattices of its {implicative} sub-varieties {being exactly varieties of De Morgan algebras/lattices}, all being disjunctive, and merely/"both all and" disjunctive|implicative sub-quasi-varieties, /"disjunctive|implicative ones" appearing to be varieties.

1. Introduction

According to [18]/[17], an/a *implicative/disjunctive system for* a class of algebras is a finite/[finite] set of quaternary equations defining implication/disjunction of two equations in each member of the class, the quasi-/pre-variety /(viz., abstract hereditary multiplicative class; cf. [20]) generated by this being called */[finitely] implicative/disjunctive* therein/here. On the other hand, implicative varieties appear exactly semi-simple ones with REDPC in the sense of [5] proved therein exactly (subdirectly, in our extended terminology) filtral ones, i.e., subdirectly ideal

2020 *Mathematics Subject Classification.* 06A12, 06D05, 06D15, 06D30, 08B05, 08B15, 08B26, 08C15.

Key words and phrases. congruence, filter, simple algebra, semi-lattice, distributive lattice, Stone algebra, De Morgan lattice.

ones. And what is more, direct ideality is equivalent to EDPC admitting existential parameters in definining equations. These points make rather acute the issue of admitting parameters in implicative systems, those without parameters being called *restricted* ones. Then, a [quasi-]variety is said to be *(restricted) implicative*, if its [relatively] subdirectly-irreducibles have a common (restricted) implicative system (i.e., is implicative in the sense of [18]). In this connection, the main result of the work is equivalence of (restricted) implicativity of a [quasi-]variety Q to its [relative] (sub)direct filtrality, i.e., filtrality of [Q-relative] congruences of (sub)direct products of [relatively] subdirectly-irreducibles of Q (in the equational case meaning exactly the filtrality in the sense of [5]). It is remarkable that this is proved directly {without involving extra links like EDPC and ideality} as well as uniformly for both the parameterized and restricted case. Such uniformity equally concerns proving [relative] (sub)direct filtrality of Q in case of both its [relatively] (sub)direct congruence-distributivity {i.e., distributivity of the lattices of [Q-relative] congruences of (sub)direct products of [relatively] subdirectly-irreducibles of Q (equivalent to its [relative] congruence-distributivity)}, its [relative] semi-simplicity and (universal) first-order axiomatizability of the class of [relatively] subdirectly-irreducibles of Q, but not the converse. More precisely, while the [relative] direct congruence-distributivity of [relatively] directly filtral [quasi-]varieties ensues from the distributivity of the filter lattices, the [relative] congruence-distributivity of restricted implicative [quasi-]varieties results from their disjunctivity, being due to [18, Remark 2.4], and the [relative] congruence-distributivity of disjunctive [pre-]varieties proved here, though both facts are based upon distributivity of closure systems with disjunctive bases explored here, the latter one making equally acute the problems of studying both disjunctivity and its connections with implicativity. A one more peculiarity of the restricted case consists in equivalence of restricted implicativity to semi-simplicity jointly with REDPC, while EDPC and semi-simplicity just imply implicativity, whereas the truth of the converse remains an open problem.

In view of the universal first-order axiomatizability of abstract hereditary local subclasses of locally-finite quasi-varieties ensuing from [4, Corollary 2.3], any locally-finite [quasi-]variety is then restricted implicative iff it is [relatively] both semi-simple and congruence-distributive with hereditary class of its [relatively] subdirectly-irreducibles. In particular, the variety generated by a finite set of finite lattice expansions without non-simple non-one-element subalgebras, being thus restricted implicative, taking [7] into account, has REDPC. This provides an immediate (though far from being constructive) proof/insight to/into REDPC for the semi-simple finitely-generated variety of distributive|"De Morgan" lattices|algebras, originally being due to [6]|[19]. On the other hand, a generic constructive approach to implicative systems for lattice expansions, being underlying algebras of logical matrices with prime filter truth predicate, equality determinant and equational implication, covering the varieties of [bounded] distributive lattices as well as Kleene lattices/algebras, has been due to [17, Theorems 10, 12(iii)⇒(i) and Lemma 11] jointly with [18, Lemma A.2]. However, the varieties of /[Boolean] De Morgan lattices/algebras have proved beyond it because of related negative results of [17] (cf. the paragraph followed by Example 10 therein). Concluding this work, we propose a supplementary generic constructive approach, based upon the notion of equality determinant, equally covering the mentioned three varieties.

Finally, we exemplify our general elaboration by applying it to the disjunctive non-implicative [quasi-]equational join (viz., the [quasi-]variety generated by the union) of the varieties of Stone algebras and De Morgan algebras/lattices as well as finding the lattices of its {implicative} sub-varieties {being exactly varieties of De Morgan algebras/lattices}, all being disjunctive, and merely/"both all and" disjunctive|implicative sub-quasi-varieties, /"disjunctive|implicative ones" appearing to be varieties.

The rest of the work is as follows. Section 2 is a concise summary of basic set-theoretical and algebraic issues underlying the work. Then, in Section 3 we recall preliminary issues concerning abstract disjunctivity as well as both finite locality and local finiteness. Next, Section 4 is a collection of main results of the work. Further, Section 5 exemplifies our universal elaboration within the framework of the equational join of De Morgan and Stone algebras uniformly covering both these. Finally, in Section 6 we briefly discuss principal problems remained open here.

2. General background

2.1. **Set-theoretical background.** Non-negative integers are identified with the sets/ordinals of lesser ones, "their set/ordinal"|"the ordinal‖set class" being denoted by $\omega|(\infty\|\Upsilon)$. Unless any confusion is possible, one-element sets are identified with their elements.

For any sets A, B and D as well as $\theta \subseteq A^2$, $g : A^2 \to A$, $e : A \to D$ and $h : A \to B$, let $\wp_{[K]}((B,)A)$ be the set of all subsets of A (including B) [of cardinality in $K \subseteq \infty$], $((\Delta_A|\nu_\theta)\|(A/\theta)\|\chi^B_A) \triangleq (\{\langle a, a|\theta[\{a\}]\rangle \mid a \in A\}\|\nu_\theta[A]\|(((A \cap B) \times \{1\}) \cup ((A \setminus B) \times \{0\})))$, $A^{*|+} \triangleq (\bigcup_{m\in(\omega\setminus(0|1))} A^m)$, $g_+ : A^+ \to A, \langle[\langle a, b\rangle,]c\rangle \mapsto [g]([g_+(\langle a, b\rangle),]c)$, $h^{|(/-1)}_{n|\wp} : (A^n|\wp(A/B)) \to (B^n|\wp(B/A)), (f|X) \mapsto ((f \circ h)|h^{/-1}[X])$ with $n \in \omega$, $(h \times e) : A \to (B \times D), a \mapsto \langle h(a), e(a)\rangle$ and $\varepsilon_B : (\Upsilon^B)^2 \to \wp(B), \langle d, e\rangle \mapsto \{b \in B \mid \pi_b(d) = \pi_b(e)\}$, A-tuples {viz., functions with domain A} being written in the sequence form $\bar{t}$ with t_a, where $a \in A$, standing for $\pi_a(\bar{t})$. Then, for any $(\bar{a}|C) \in (A^*|\wp(A))$, by induction on the length (viz., domain) of any $\bar{b} = \langle[\bar{c}, d]\rangle \in A^*$, put $((\bar{a} * \bar{b})|(\bar{b}(\cap/\setminus)C)) \triangleq ((([\langle]\bar{a}[*\bar{c}, d\rangle])|(\langle[\bar{c}(\cap/\setminus)C(, d)]\rangle))\|[(\text{provided } d \in / \notin C)]$. Likewise, given any $\overline{S} \in \Upsilon^A$ and $\bar{f} \in \prod_{a\in A} S^B_a$, let $(\prod \bar{f}) : B \to (\prod_{a\in A} S_a), b \mapsto \langle f_a(b)\rangle_{a\in A}$, in which case

$$\ker(\prod \bar{f}) \quad = \quad (B^2 \cap (\bigcap_{a\in A} (\ker f_a))), \tag{2.1}$$

$$\forall a \in A : f_a \quad = \quad ((\prod \bar{f}) \circ \pi_a), \tag{2.2}$$

$f_0 \times f_1$ standing for $(\prod \bar{f})$, whenever $A = 2$.

An $X \in Y \subseteq \wp(A)$ is said to be *[K-]meet-irreducible/maximal in* Y [where $K \subseteq \infty$]/, if $\forall Z \in (\wp_{[K]/\{1\}}(Y) : (((A/X) \cap (\bigcap Z)) = X) \Rightarrow (X \in Z)$ with their set denoted by $(\mathrm{MI}^{[K]} / \max)(Y)$, "finitely-" standing for "ω-" within any related context. Next, a $\mathcal{U} \subseteq \wp(A)$ is said to be *upward-directed*, if $\forall \mathcal{S} \in \wp_\omega(\mathcal{U}) : \exists T \in (\mathcal{U} \cap \wp(\bigcup \mathcal{S}, A))$, subsets of $\wp(A)$ closed under unions of upward directed subsets being called *inductive.* Further, a *[finitary] closure operator over* A is any unary operation on $\wp(A)$ with $\forall X \in \wp(A), \forall Y \in \wp(X) : (X \cup C(C(X)) \cup C(Y)) \subseteq C(X)[= (\bigcup C[\wp_\omega(X)])]$. Finally, a *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ containing A and closed under intersections of subsets containing A, any $\mathcal{B} \subseteq \mathcal{C}$ with $\mathcal{C} = \{A \cap (\bigcap \mathcal{S}) \mid \mathcal{S} \subseteq \mathcal{B}\}$ being called a *(closure) basis of* $\mathcal{C}$ and determining the closure operator $C_\mathcal{B} \triangleq \{\langle Z, A \cap (\bigcap(\mathcal{X} \cap \wp(Z, A)))\rangle \mid Z \in \wp(A)\}$ over A with $(\operatorname{img} C_\mathcal{B}) = \mathcal{C}$. Conversely, $\operatorname{img} C$ is a closure system over A such that $C_{\operatorname{img} C} = C$, being inductive iff C is finitary, and forming a complete lattice under the partial ordering by inclusion with meet/join $(\Delta_{\wp(A)}/C)(A\cap((\bigcap / \bigcup)\mathcal{S}))$ of any $\mathcal{S} \subseteq (\operatorname{img} C)$, C and $\operatorname{img} C$ being called *dual to* one another.

Remark 2.1. Due to Zorn Lemma, according to which any non-empty inductive set has a maximal element, $MI^{[K]}(\mathcal{C})$ is a basis of any inductive closure system $\mathcal{C}$. $\square$

A *filter/ideal on* A is any $\mathcal{F} \subseteq \wp(A)$ such that, for all $\mathcal{S} \in \wp_\omega(\wp(A))$, $(\mathcal{S} \subseteq \mathcal{F}) \Leftrightarrow ((A \cap ((\bigcap/\bigcup)\mathcal{S})) \in \mathcal{F})$ "the set $\mathrm{Fi}(A)$ of them being an inductive closure system over $\wp(A)$ with dual closure operator (of filter generation) Fg_A such that $\mathrm{Fg}_A(\mathcal{S}) = \wp(A \cap (\bigcap \mathcal{S}), A)$"/. Then, an *ultra-filter on* A is any filter $\mathcal{U}$ on A such that $\wp(A) \setminus \mathcal{U}$ is an ideal on A.

2.2. **Algebraic background.** Unless otherwise specified, we deal with a fixed but arbitrary finitary functional signature Σ, Σ-algebras/"their carriers" being denoted by same capital Fraktur/Italic letters (with same indices, if any) "with denoting their class by A_Σ"/. Given any $\alpha \in (\infty \setminus 1)$, let $\mathrm{Tm}^\alpha_\Sigma$ be the carrier of the absolutely-free Σ-algebra $\mathfrak{Tm}^\alpha_\Sigma$, freely-generated by the set $V_\alpha \triangleq \{x_\beta\}_{\beta\in\alpha}$ of *(first α) variables*, and $\mathrm{Eq}^\alpha_\Sigma \triangleq (\mathrm{Tm}^\alpha_\Sigma)^2$, $\phi \approx /(\lessapprox \mid \gtrapprox)\psi$, where $\phi, \psi \in \mathrm{Tm}^\alpha_\Sigma$ /"and $\Sigma_+ \triangleq \{\wedge, \vee\} \ni \wedge \in \Sigma$", meaning $\langle \phi/(\phi \wedge \psi), \psi/(\phi|\psi)\rangle$ "and being called a Σ-*equation of rank* α with denoting the set of variables actually occurring in it by $\mathrm{Var}(\phi \approx \psi) \in \wp_\omega(V_\alpha)$"/. /"Likewise, for any Σ-algebra $\mathfrak{A}$ and $a, b \in A$, $(a(\leqslant \mid \geqslant)^{\mathfrak{A}} b)\|[a,b]_{\mathfrak{A}}$ stands for $((a|b) = (a \wedge^{\mathfrak{A}} b))\|\{c \in A \mid a \leqslant^{\mathfrak{A}} c \leqslant^{\mathfrak{A}} b\}$." Then, any $\langle \Gamma, \Psi\rangle \in (\wp_{\infty/(1[\cup\omega])}(\mathrm{Eq}^\alpha_\Sigma) \times \mathrm{Eq}^\alpha_\Sigma)$ /"with $\alpha \in \omega$" is called a Σ-*implication/-[quasi-]identity of rank* α, written as $\Gamma \to \Psi$ /[and identified with Ψ] as well as treated as the universal infinitary/first-order /[positive] strict Horn sentence $\forall_{\beta\in\alpha} x_\beta((\bigwedge \Gamma) \to \Psi)$.

Subclasses of A_Σ closed under $\mathbf{I}/\mathbf{S}/\mathbf{P}^{[\mathrm{U}|\mathrm{SD}]}$ are referred to as *abstract/hereditary/ [ultra-|sub-]multiplicative* (cf. [11]). Given a $\mathsf{K} \subseteq \mathsf{A}_\Sigma \ni \mathfrak{A}$ and a $K \subseteq \infty$, set $\mathsf{K}_K \triangleq \{\mathfrak{B} \in \mathsf{K} \mid |B| \in K\}$, $\hom(\mathfrak{A}, \mathsf{K}) \triangleq (\bigcup\{\hom(\mathfrak{A}, \mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}\}$ and $\mathrm{Co}_{\mathsf{K}}(\mathfrak{A}) \triangleq \{\theta \in \mathrm{Co}(\mathfrak{A}) \mid (\mathfrak{A}/\theta) \in \mathsf{K}\}$, whose elements are called K-*(relative)congruences of* $\mathfrak{A}$, in which case, by the Homomorphism Theorem:

$$\ker[\hom(\mathfrak{A}, \mathsf{K})] = \mathrm{Co}_{\mathbf{IS}\mathsf{K}}(\mathfrak{A}). \tag{2.3}$$

Furthermore, for any set I, any $\overline{\mathfrak{B}} \in \mathsf{A}^I_\Sigma$ and any $\bar{h} \in (\prod_{i\in I} \hom(\mathfrak{A}, \mathfrak{B}_i))$:

$$(\prod \bar{h}) \in \hom(\mathfrak{A}, \prod_{i\in I} \mathfrak{B}_i). \tag{2.4}$$

Remark 2.2. As, for any $\mathfrak{A} \in \mathsf{A}_\Sigma$, by the Homomorphism Theorem, $\forall \Theta \subseteq \mathrm{Co}(\mathfrak{A})$: $\theta \triangleq (A^2 \cap (\bigcap \Theta)) \in \mathrm{Co}(\mathfrak{A}), \bar{h}^\Theta \triangleq \langle \nu_\theta^{-1} \circ \nu_\vartheta\rangle_{\vartheta\in\Theta} \in (\prod_{\vartheta\in\Theta} \hom(\mathfrak{A}/\theta, \mathfrak{A}/\theta), h \triangleq (\prod \bar{h}^\Theta) \in \hom(\mathfrak{A}/\theta, \prod_{\vartheta\in\Theta}(\mathfrak{A}/\theta)), (\ker h) = \Delta_{A/\theta}, \forall \vartheta \in \Theta : \pi_\vartheta[h[A/\theta]] = (A/\vartheta)$, in view of (2.1), (2.2) and (2.4), while $\forall I \in \Upsilon, \forall \overline{\mathfrak{B}} \in \mathsf{A}^I_\Sigma, \forall \theta \in \mathrm{Co}(\mathfrak{A}), \forall h \in \hom(\mathfrak{A}/\theta, \prod_{i\in I} \mathfrak{B}_i) : (((\ker h) = \Delta_{A/\theta})\&(\forall i \in I : \pi_i[h[A/\theta]] = B_i)) \Rightarrow ((\forall i \in I : \theta_i \triangleq \ker((\nu_\theta \circ h) \circ \pi_i) \in \mathrm{Co}(\mathfrak{A}), h_i \triangleq (\nu_\theta^{-1} \circ ((\nu_\theta \circ h) \circ \pi_i)) \in \hom(\mathfrak{A}/\theta_i, \mathfrak{B}_i), (\ker h_i) = \Delta_{A/\theta_i}, h_i[A/\theta_i] = B_i)\&(\theta = (A^2 \cap (\bigcap_{i\in I} \theta_i))))$, $\mathrm{Co}_{\mathsf{K}}(\mathfrak{A})$, where $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, being a basis of the closure system $\mathrm{Co}_{\mathbf{IP}^{\mathrm{SD}}\mathsf{K}}(\mathfrak{A})$ over A^2, if K is abstract, is a closure system over A^2 iff K is both abstract and sub-multiplicative, and, by (2.3) and the bijectivity of $\nu_{\Delta_A} \in \hom(\mathfrak{A}, \mathfrak{A}/\Delta_A)$, $\mathfrak{A} \in \mathbf{ISP}\mathsf{K}(= \mathbf{IP}^{\mathrm{SD}}[\mathbf{I}]\mathbf{S}\mathsf{K})$ iff $(A^2 \cap (\bigcap \ker[\hom(\mathfrak{A}, \mathsf{K})])) = \Delta_A$. □

Thus, [providing a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ is both abstract and sub-multiplicative] the closure operator (of [K-]congruence generation) over A^2 dual to the closure system $\mathrm{Co}_{[\mathsf{K}]}(\mathfrak{A})$ over A^2 is denoted by $\mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{K}]}$.

According to [20], *pre-varieties* are abstract hereditary multiplicative subclasses of A_Σ (these are exactly model classes of theories constituted by Σ-implications of unlimited rank, and so are said to be *implicative/implicational*; cf., e.g., [3]/[14])), $\mathbf{ISP}\mathsf{K} = \mathbf{IP}^{\mathrm{SD}}\mathbf{S}\mathsf{K}$ being the least one including and so called *generated by* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$. Then, *[quasi-]varieties* are [ultra-multiplicative] pre-varieties closed under $\mathbf{H}^{[\mathrm{I}]}[\triangleq \mathbf{I}]$ (these are exactly model classes of sets of Σ-[quasi-]identities of unlimited finite rank, and so are said to be *[quasi-]equational*; cf., e.g., [11]), $\mathbf{H}^{[\mathrm{I}]}\mathbf{SP}[\mathbf{P}^{\mathrm{U}}]\mathsf{K}$ being the least one including and so called *generated by* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$. In particular, given a quasi-variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma \ni \mathfrak{A}$ axiomatized by a set $\mathfrak{Q}$ of Σ-quasi-identities of finite rank and a $\theta \in \mathrm{Co}(\mathfrak{A})$, $\hom(\mathfrak{Tm}^\omega_\Sigma, \mathfrak{A}) = \{h \circ \nu_\theta \mid h \in \hom(\mathfrak{Tm}^\omega_\Sigma, \mathfrak{A}/\theta)\}$, in which case θ is Q-relative iff $\forall h \in \hom(\mathfrak{Tm}^\omega_\Sigma, \mathfrak{A}), \forall(\Gamma \to \Phi) \in \mathfrak{Q} : (h_2[\Gamma] \subseteq \theta) \Rightarrow (h_2(\Phi))$, and so $\mathrm{Co}_\mathsf{Q}(\mathfrak{A})$ is inductive, for $\mathrm{Co}_\mathsf{Q}(\mathfrak{A})$ is so, as Σ is finitary.

Given a [pre-]variety $\mathsf{P} \subseteq \mathsf{A}_\Sigma$, an $\mathfrak{A} \in \mathsf{P}$ is called *[*P*-{relatively }]simple/⟨*$K-$*⟩subdirectly-irreducible* ⟨where $K \subseteq \infty$⟩, if $\Delta_A \in (\max / MI^{\langle K\rangle})(\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{A}) \setminus (\{A^2\}/\varnothing))$, in which case $|A| \neq 1$, the class of them being denoted by $(\mathrm{Si} / \mathrm{SI}^{\langle K\rangle})_{[\mathsf{P}]}(\mathsf{P})$. Then, P is said to be *[relatively] ⟨*$K-$*⟩semi-simple/subdirectly-representable*, if $(\mathrm{SI}^{\langle K\rangle}_{[\mathsf{P}]}(\mathsf{P})/\mathsf{P}) \subseteq | = (\mathrm{Si}_{[\mathsf{P}]}(\mathsf{P})/\mathbf{IP}^{\mathrm{SD}}\, \mathrm{SI}^{\langle K\rangle}_{[\mathsf{P}]}(\mathsf{P}))$. Likewise, it is said to be *[relatively] {(sub)directly} congruence-distributive*, if, for each $\mathfrak{A} \in (\mathsf{P}\{\cap \mathbf{P}^{(\mathrm{SD})}\, \mathrm{SI}_{[\mathsf{P}]}(\mathsf{P})\})$, $\mathrm{Co}_{[\mathsf{P}]}$ $(\mathfrak{A})$ is distributive.

Lemma 2.3. *Let* $\mathfrak{A}, \mathfrak{B} \in \mathsf{A}_\Sigma[\supseteq \mathsf{K} \supseteq ((\mathbf{IS})/\mathbf{I})\mathsf{K}]$ *and* $h \in \hom(\mathfrak{A}, \mathfrak{B})$ */"with* $h[A] = B$*". Then,* $(h_2)^{-1}_\wp \restriction \mathrm{Co}_{[\mathsf{K}]}(\mathfrak{B})$ */"and* $(h_2)_\wp \restriction (\mathrm{Co}_{[\mathsf{K}]}(\mathfrak{A}) \cap \wp(\ker h, A^2))$*" is/are a/ homomorphism/"inverse to one another isomorphisms" from/between the poset* $\mathrm{Co}_{[\mathsf{K}]}(\mathfrak{B})$ *to/and that* $\mathrm{Co}_{[\mathsf{K}]}(\mathfrak{A}) \cap \wp(\ker h, A^2)$ *under the partial ordering by* $\subseteq$*.*

Proof. The []-non-optional part is well-known. [Then, for any $\theta \in \mathrm{Co}(\mathfrak{B})$, $g \triangleq (h \circ \nu_\theta) \in \hom(\mathfrak{A}, \mathfrak{B}/\theta)$ /"with $g[A] = (B/\theta)$", while $\vartheta \triangleq (\ker g) = h_2^{-1}[\theta] \in (\mathrm{Co}(\mathfrak{A}) \cap \wp(\ker h, A^2))$, in which case, by the Homomorphism Theorem, $\nu_\vartheta^{-1} \circ g$ is an embedding/isomorphism from $\mathfrak{A}/\vartheta$ into/onto $\mathfrak{B}/\theta$, and so $(\vartheta \in \mathrm{Co}_\mathsf{K}(\mathfrak{A})) \Leftarrow / \Leftrightarrow (\theta \in \mathrm{Co}_\mathsf{K}(\mathfrak{B}))$, as required.] ☐

This immediately yields:

Corollary 2.4. *For any [pre-]variety* $\mathsf{P} \subseteq \mathsf{A}_\Sigma$*,* $(\mathrm{SI}^{(K)} \mid \mathrm{Si})_{[\mathsf{P}]}(\mathsf{P})$ *is abstract (where* $K \subseteq \infty$*)|.*

Corollary 2.5. *Any [quasi-]variety* Q *is [relatively] (*K*-)subdirectly representable (where* $K \subseteq \infty$*). In particular, it is [relatively] subdirectly congruence-distributive iff it is [relatively] congruence-distributive.*

Proof. For any $\mathfrak{A} \in \mathsf{Q}$, by Lemma 2.3, $\mathrm{MI}^{(K)}(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{A})) = \mathrm{Co}_{\mathrm{SI}^{(K)}_{[\mathsf{Q}]}(\mathsf{Q})}(\mathfrak{A})$, the former/latter being a basis of $\mathrm{Co}_\mathsf{Q}(\mathfrak{A}) / \mathrm{Co}_{\mathbf{IP}^{\mathrm{SD}}\, \mathrm{SI}^{(K)}_{[\mathsf{Q}]}(\mathsf{Q})}(\mathfrak{A})$, in view of Remark 2.1/"2.2 and Corollary 2.4", in which case these are equal, and so $\mathfrak{A} \in \mathbf{IP}^{\mathrm{SD}}\, \mathrm{SI}^{(K)}_{[\mathsf{Q}]}(\mathsf{Q})$, for $\nu_{\Delta_A} \in \hom(\mathfrak{A}, \mathfrak{A}/\Delta_A)$ is bijective. Then, Lemma 2.3 completes the argument. ☐

Corollary 2.6. *Let* $\mathfrak{A}, \mathfrak{B} \in \mathsf{A}_\Sigma[\supseteq \mathsf{K} \supseteq \mathbf{IP}^{\mathrm{SD}}\mathsf{K}]$ *and* $h \in \hom(\mathfrak{A}, \mathfrak{B})$ *with* $h[A] = B$*. Then, for all* $X \subseteq A^2$*,* $\mathrm{Cg}^\mathfrak{A}_{[\mathsf{K}]}(X \cup (\ker h)) = h_2^{-1}[\mathrm{Cg}^\mathfrak{B}_{[\mathsf{K}]}(h_2[X])]$*.*

Proof. By Lemma 2.3, we have:

$$h_2^{-1}[\mathrm{Cg}^\mathfrak{B}_{[\mathsf{K}]}(h_2[X])] = h_2^{-1}[B^2 \cap (\bigcap\{\theta \in \mathrm{Co}_{[\mathsf{K}]}(\mathfrak{B}) \mid h_2[X] \subseteq \theta\})] =$$
$$(h_2^{-1}[B^2] \cap (\bigcap\{h_2^{-1}[\theta] \mid \theta \in \mathrm{Co}_{[\mathsf{K}]}(\mathfrak{B}), X \subseteq h_2^{-1}[\theta]\})) =$$
$$(A^2 \cap (\bigcap\{\vartheta \in \mathrm{Co}_{[\mathsf{K}]}(\mathfrak{B}) \mid (X \cup (\ker h)) \subseteq \vartheta\})) = \mathrm{Cg}^\mathfrak{A}_{[\mathsf{K}]}(X \cup (\ker h)).$$ ☐

2.2.1. *Filtral congruences.* Let I be a set, $\mathcal{F}$ a{n ultra-}filter on I [$\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ a quasi-variety] and $\overline{\mathfrak{A}} \in (\mathsf{A}_\Sigma[\cap \mathsf{Q}])^I$ as well as $\mathfrak{B}$ a subalgebra of its direct product. Then, by Lemma 2.3, for each $i \in I$, $(B^2 \cap (\ker \pi_i)) = ((\pi_i \restriction B)_2^{-1}[\Delta_{A_i}] \in \mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$, as $(\pi_i \restriction B) \in \hom(\mathfrak{B}, \mathfrak{A}_i)$ and $\Delta_{A_i} \in \mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{A}_i)$, in which case, for all $K \subseteq J \subseteq I$, the closure system $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ on B^2 contains $\theta^B_J \triangleq (B^2 \cap \varepsilon_I^{-1}[\wp(J, I)]) = (B^2 \cap (\bigcap_{j \in J} \ker \pi_j)) \subseteq \theta^B_K$, $\Theta^B_\mathcal{F} \triangleq \{\theta^B_L \mid L \in \mathcal{F}\}$ being then upward-directed, and so

$\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$, being inductive, contains $\theta^B_{\mathcal{F}} \triangleq (\bigcup \Theta^B_{\mathcal{F}}) = (B^2 \cap \varepsilon_I^{-1}[\mathcal{F}])$, called *($\mathcal{F}$-){ultra-}filtral*. Clearly, for any $\mathcal{X} \subseteq \mathrm{Fi}(I)$ |"with $(\bigcup \mathcal{X}) \in \mathrm{Fi}(I)$",

$$\theta^B_{\wp(I)\cap((\bigcap|\bigcup)\mathcal{X})} = (B^2 \cap ((\bigcap|\bigcup)\{\theta^B_{\mathcal{F}} \mid \mathcal{F} \in \mathcal{X}\})). \tag{2.5}$$

A [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ is said to be *[relatively] {(sub)directly} filtral*, if every [Q-]congruence of each member of $\mathbf{SP}\,\mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})\{\cap\mathbf{P}^{(\mathrm{SD})}\,\mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})\}$ is filtral {(cf. [5])}, in which case, by Corollary 2.4, Q is [relatively] semi-simple, because any $\mathfrak{C} \in \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})$ is isomorphic to $\mathfrak{D} \triangleq \mathfrak{C}^1$, while $\mathrm{Fi}(1) = \{1, 2\}$, whereas $\theta^D_1 = \Delta_D \neq D^2 = \theta^D_2$, for $|D| = |C| \neq 1$, and so $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{D}) = \{\Delta_D, D^2\}$, i.e., $\mathfrak{D}$ is [Q-]simple, viz., $\mathfrak{C}$ is so.

Lemma 2.7 (cf. [7] for the []()-non-optional case)**.** *Let Q be a [quasi-]variety, I a set, $\overline{\mathfrak{A}} \in \mathsf{Q}^I$, $\mathfrak{B} \in \mathbf{S}(\prod \overline{\mathfrak{A}})$ and $\theta \in \mathrm{MI}^{(\omega)}(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}))$. Suppose $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ is distributive. Then, there is an ultra-filter $\mathcal{U}$ on I such that $\theta^B_{\mathcal{U}} \subseteq \theta$.*

Proof. By (2.5), $S \triangleq \{\mathcal{F} \in \mathrm{Fi}(I) \mid \theta^B_{\mathcal{F}} \subseteq \theta\}$ is inductive, for $\mathrm{Fi}(I)$ is so, in which case, by Zorn Lemma, it has a maximal element $\mathcal{U}$, and so, for any $\mathcal{X} \in \wp_\omega(\wp(I))$ such that $Y \triangleq (\bigcup \mathcal{X}) \in \mathcal{U}$, $(\mathcal{X} \cap \mathcal{U}) \neq \varnothing$, as, for each $Z \in \mathcal{X}$, $\theta^B_{\mathcal{F}_Z} \in \mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ with $\mathcal{U} \subseteq \mathcal{F}_Z \triangleq \mathrm{Fg}_I(\mathcal{U} \cup \{Z\}) \in \mathrm{Fi}(I)$, while $\mathcal{U} = \mathrm{Fg}_I(\mathcal{U}) = \mathrm{Fg}_I(\mathcal{U} \cup \{Y\}) = (\wp(I) \cap (\bigcap\{\mathcal{F}_Z \mid Z \in \mathcal{X}\}))$, whereas, by (2.5), $\theta = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup \theta^B_{\mathcal{U}}) = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup (B^2 \cap (\bigcap\{\theta^B_{\mathcal{F}_Z} \mid Z \in \mathcal{X}\}))) = (B^2 \cap (\bigcap\{\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup \theta^B_{\mathcal{F}_Z}) \mid Z \in \mathcal{X}\}))$, that is, for some $Z \in \mathcal{X}$, $\theta = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\theta \cup \theta^B_{\mathcal{F}_Z}) \supseteq \theta^B_{\mathcal{F}_Z}$, i.e., $\mathcal{U} \subseteq \mathcal{F}_Z \in S$, viz., $Z \in \mathcal{F}_Z = \mathcal{U}$, as required. □

This, by Lemma 2.3 and the Homomorphism Theorem {as well as [4, Corollary 2.3]}, yields:

Corollary 2.8. *Let K be a {finite} class of {finite} Σ-algebras and $\mathsf{P} \triangleq \mathbf{H}^{(\mathrm{I})}\mathbf{SP}\mathsf{K}$. Suppose P is a [relatively] congruence-distributive [quasi-]variety. Then, $\mathrm{SI}^{\langle\omega\rangle}_{[\mathsf{P}]}(\mathsf{P}) \subseteq \mathbf{H}^{(\mathrm{I})}\mathbf{SP}^{\mathrm{U}}\mathsf{K}\{\subseteq \mathbf{H}^{(\mathrm{I})}\mathbf{S}\mathsf{K}$, in which case its members are finite, and so $\mathrm{SI}^{\omega}_{[\mathsf{P}]}(\mathsf{P}) = \mathrm{SI}_{[\mathsf{P}]}(\mathsf{P})\}$.*

2.2.2. *Subdirect products versus subalgebras.*

Lemma 2.9 (cf. [8])**.** *Let $\mathfrak{A} \in \mathsf{A}_\Sigma$ and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Then, $h^B_A \triangleq \{\langle\bar{a}, b\rangle \in (A^\omega \times B) \mid |\omega \setminus \varepsilon_\omega(\bar{a}, \omega \times \{b\})| \in \omega\} \supseteq (\bigcup\{\{\langle\omega \times \{b\}, b\rangle\} \cup \{\langle((\omega \setminus \{i\}) \times \{b\}) \cup \{\langle i, a\rangle\}, b\rangle \mid i \in \omega, a \in A\} \mid b \in B\})$ is a function forming a subalgebra of $\mathfrak{A}^\omega \times \mathfrak{B}$, in which case it is a surjective homomorphism from $\mathfrak{C}^B_A \triangleq (\mathfrak{A}^\omega{\upharpoonright}(\mathrm{dom}\, h^B_A))$ onto $\mathfrak{B}$, and so $\mathfrak{C}^B_A$ is a subdirect product of $\omega \times \{\mathfrak{A}\}$.*

Corollary 2.10. *Let $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ be a [relatively] subdirectly filtral [quasi-]variety. Then, $\mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{I}(\prod \varnothing)$ is hereditary.*

Proof. Let $\mathfrak{A} \in (\mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{I}(\prod \varnothing))$ and $\mathfrak{B}$ a non-one-element subalgebra of $\mathfrak{A}$, in which case $|A| \neq 1$, and so, by Lemma 2.9, $h \triangleq h^B_A$ is a surjective homomorphism from the subdirect product $\mathfrak{C} \triangleq \mathfrak{C}^B_A$ of $(\omega \times \{\mathfrak{A}\}) \in \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})^\omega$ onto $\mathfrak{B}$. Consider any $\theta \in (\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{\Delta_B\})$ and take any $\langle a, b\rangle \in (\theta \setminus \Delta_B) \neq \varnothing$, in which case, by Lemma 2.3, $\vartheta \triangleq h_2^{-1}[\theta] \in \mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{C})$, while $\theta = h_2[\vartheta]$, whereas $(\bar{c}|\bar{d}) \triangleq (\omega \times \{a|b\}) \in h^{-1}[\{a|b\}] \subseteq C$, and so there is some $\mathcal{F} \in \mathrm{Fi}(\omega)$ such that $\langle\bar{c}, \bar{d}\rangle \in \vartheta = \theta^C_{\mathcal{F}}$, while, as $a \neq b$, $\varnothing = \varepsilon_\omega(\bar{c}, \bar{d}) \in \mathcal{F}$, whereas $\mathcal{F} = \wp(\omega)$. Then, $\vartheta = C^2$, in which case $\theta = h_2[C^2] = B^2$, and so $\mathfrak{B} \in \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})$, as required. □

3. Preliminaries

3.1. **Closure systems with disjunctive closure bases.** Let A be a set, C a closure operator over it, $\delta : A^2 \to \wp(A)$ and $\forall X, Y \subseteq A : \delta(X, Y) \triangleq (\bigcup \delta[X \times Y])$.

Then, an $X \subseteq A$ is said to be *weakly/ δ-disjunctive*, if $\forall a, b \in A : ((\{a, b\} \cap X) \neq \varnothing) \Rightarrow / \Leftrightarrow (\delta(a, b) \subseteq X)$, in which case:

$$\forall Y, Z \in \wp(A) : ((Y \nsubseteq X) \Rightarrow (Z \subseteq X)) \Rightarrow / \Leftrightarrow (\delta(Y, Z) \subseteq X) \tag{3.1}$$

/"the set of those in an $\mathcal{S} \subseteq \wp(A)$ being denoted by $\mathcal{S}_{\mho}$". Likewise, C is said to be *weakly/ [multiply] δ-disjunctive*, if $\forall X, Y \in \wp_{\{1\}[\cup\infty]}(A), \forall Z \in \wp(A) : C(\delta(X, Y) \cup Z) \subseteq / = (C(X \cup Z) \cap C(Y \cup Z))$ /[in which case $\forall X, Y, Z \in (\operatorname{img} C) : (C(X \cup Z) \cap C(Y \cup Z)) = C(\delta(X, Y) \cup Z) = C(C(\delta(X, Y)) \cup Z) = C((C(X) \cap C(Y)) \cup Z) = C((X \cap Y) \cup Z)$, and so $\operatorname{img} C$ is distributive].

Lemma 3.1. *Let $\mathcal{B} \subseteq \wp(A)_{\mho}$. Then, $C_{\mathcal{B}}$ is multiply δ-disjunctive. In particular, the closure system* $(\operatorname{img} C_{\mathcal{B}})$ *over A with basis $\mathcal{B}$ is distributive.*

Proof. By (3.1), we have:

$$\begin{aligned}
&\forall X, Y, Z \in \wp(A), \forall a \in A : (a \in (C_{\mathcal{B}}(X \cup Z) \cap C_{\mathcal{B}}(Y \cup Z))) \\
&\qquad \Leftrightarrow (\forall W \in \mathcal{B} : (((X \subseteq W)\&(Z \subseteq W)) \Rightarrow (a \in W)) \\
&\qquad\qquad \&(((Y \subseteq W)\&(Z \subseteq W)) \Rightarrow (a \in W))) \\
&\qquad \Leftrightarrow (\forall W \in \mathcal{B} : (((X \nsubseteq W) \Rightarrow (Y \subseteq W))\&(Z \subseteq W)) \Rightarrow (a \in W)) \\
&\qquad \Leftrightarrow (\forall W \in \mathcal{B} : ((\delta(X, Y) \subseteq W)\&(Z \subseteq W)) \Rightarrow (a \in W)) \\
&\qquad \Leftrightarrow (a \in C_{\mathcal{B}}(\delta(X, Y) \cup Z)). \quad \square
\end{aligned}$$

Lemma 3.2. *Suppose C is weakly/ δ-disjunctive. Then,*

$$(\operatorname{img} C)_{\mho} \subseteq / = (\mathrm{MI}^{\omega}(\operatorname{img} C) \cup \{A\}).$$

Proof. Clearly, $A \in (\operatorname{img} C)$ is $\mho$-disjunctive. Now, consider any $X \in \mathrm{MI}^{\omega}(\operatorname{img} C)$, in which case, providing C is δ-disjunctive, $\forall a, b \in A : (\delta(a, b) \subseteq X) \Leftrightarrow (X = C(X) = C(\delta(a, b) \cup X) = (C(\{a\} \cup X) \cap C(\{b\} \cup X))) \Leftrightarrow (X \in \{C(\{a\} \cup X), C(\{b\} \cup X)\}) \Leftrightarrow ((\{a, b\} \cap X) \neq \varnothing)$, and so X is then $\mho$-disjunctive. Finally, consider any $Y \in ((\operatorname{img} C)_{\mho} \setminus \{A\})$ and any $\mathcal{S} \in \wp_{\omega}(\operatorname{img} C)$ such that $Y = (A \cap (\bigcap \mathcal{S}))$. in which case $\mathcal{S} \neq \varnothing$, for $Y \neq A$. By induction on $n \triangleq |\mathcal{S}| \in \omega$, let us prove that $Y \in \mathcal{S}$. For take any $U \in \mathcal{S} \neq \varnothing$, in which case $\mathcal{T} \triangleq (\mathcal{S} \setminus \{U\}) \in \wp_{\omega}(\operatorname{img} C)$, $|\mathcal{T}| = (n-1) < n$, $Z \triangleq (A \cap (\bigcap \mathcal{T})) \in (\operatorname{img} C)$ and $Y = (U \cap Z)$, and so, if Y was not in $\{U, Z\} \subseteq \wp(Y, A)$, then there would be some $(c|d) \in ((U|Z) \setminus Y) \neq \varnothing$, implying $Y \neq (U \cap Z)$, because $Y \nsupseteq \delta(c, d) \subseteq C(\delta(c, d) \cup Y) \subseteq (C(\{c\} \cup Y) \cap C(\{d\} \cup Y)) \subseteq (C(U) \cap C(Z)) = (U \cap Z)$. Hence, $Y \in \{U, Z\}$, that is, either $Y = U \in \mathcal{S}$ or $Y = Z$, in which case, by induction hypothesis, $Y \in \mathcal{S}$, and so, anyway, $Y \in \mathcal{S}$. $\square$

This, by Remark 2.1 and Lemma 3.1, immediately yields:

Corollary 3.3. *If C is both δ-disjunctive and finitary, then* $(\operatorname{img} C)_{\mho}$ *is a basis of* $\operatorname{img} C$*, in which case C is multiply δ-disjunctive, and so* $\operatorname{img} C$ *is distributive.*

3.1.1. *Application to relatively directly filtral quasi-varieties.*

Lemma 3.4. Fg_A *is* $(\cup{\restriction}\wp(A)^2)$*-disjunctive, in which case the set of ultra-filters on A is a basis of* $\mathrm{Fi}(A)$*, and so this is distributive.*

Proof. Let $X, Y, Z \in \wp(A) \supseteq \mathcal{S}$. If $Z \in \mathrm{Fg}_A(\{X \cup Y\} \cup \mathcal{S})$, then there is some $\mathcal{T} \in \wp_{\omega}(\mathcal{S})$ such that $((X \cap (\bigcap \mathcal{T})) \cup (Y \cap (\bigcap \mathcal{T}))) = ((X \cup Y) \cap (\bigcap \mathcal{T})) \subseteq Z$, in which case $(X \cap (\bigcap \mathcal{T})) \subseteq Z \supseteq (Y \cap (\bigcap \mathcal{T}))$, and so $Z \in (\mathrm{Fg}_A(\{X\} \cup \mathcal{S}) \cap \mathrm{Fg}_A(\{Y\} \cup \mathcal{S})$. Conversely, if $Z \in (\mathrm{Fg}_A(\{X\} \cup \mathcal{S}) \cap \mathrm{Fg}_A(\{Y\} \cup \mathcal{S}))$, then there are some $\mathcal{U}, \mathcal{V} \in \wp_{\omega}(\mathcal{S})$ such that $(X \cap (\bigcap \mathcal{U})) \subseteq Z \supseteq (Y \cap (\bigcap \mathcal{V}))$, in which case $\mathcal{W} \triangleq (\mathcal{U} \cup \mathcal{V}) \in \wp_{\omega}(\mathcal{S})$, while $(X \cap (\bigcap \mathcal{W})) \subseteq Z \supseteq (Y \cap (\bigcap \mathcal{W}))$, that is, $Z \supseteq ((X \cap (\bigcap \mathcal{W})) \cup (Y \cap (\bigcap \mathcal{W}))) = ((X \cup Y) \cap (\bigcap \mathcal{W}))$, and so $Z \in \mathrm{Fg}_A(\{X \cup Y\} \cup \mathcal{S})$. In this way, Corollary 3.3 completes the argument. $\square$

Corollary 3.5. *Let I be a set, $\mathcal{F}, \mathcal{G} \in \mathrm{Fi}(I)$ $\{$and $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ a [relatively] directly filtral [quasi-]variety$\}$ as well as $\overline{\mathfrak{A}} \in ((\mathsf{A}_\Sigma\{\cap \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})\}) \setminus \mathbf{I}(\prod \varnothing))^I$ with its direct product $\mathfrak{B}$. Suppose $\mathcal{F} \nsubseteq \mathcal{G}$. Then, $\theta^B_{\mathcal{F}} \nsubseteq \theta^B_{\mathcal{G}}$ $\{$in which case $\{\langle \mathcal{H}, \theta^B_{\mathcal{H}} \rangle \mid \mathcal{H} \in \mathrm{Fi}(I)\}$ is an isomorphism from $\mathrm{Fi}(I)$ onto $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$, and so Q is [relatively] directly congruence-distributive$\}$.*

Proof. Take any $J \in (\mathcal{F} \setminus \mathcal{G}) \neq \varnothing$, $\bar{a} \in (\prod_{j \in J} A_j) \neq \varnothing$ and $\bar{b} \in (\prod_{i \in (I \setminus J)} (A^2_i \setminus \Delta_{A_i})) \neq \varnothing$, in which case $(\bar{c}|\bar{d}) \triangleq (\bar{a} \cup \langle \pi_{0|1}(b_i) \rangle_{i \in I}) \in B$, while $\varepsilon_I(\bar{c}, \bar{d}) = J$, and so $\langle \bar{c}, \bar{d} \rangle \in (\theta^B_{\mathcal{F}} \setminus \theta^B_{\mathcal{G}})$. $\{$Then, (2.5) and Lemma 3.4 complete the argument.$\}$ □

3.2. **Abstract hereditary local subclasses of locally-finite quasi-varieties.** A quasi-variety is said to be *locally-finite*, if every finitely-generated member of it is finite, any finitely-generated [quasi-]variety being locally-finite. Likewise, a class of Σ-algebras is said to be *(finitely-)local*, if it contains every Σ-algebra, each finitely-generated subalgebra of which is in the class, any quasi-variety being local. As an immediate consequence of [18, Lemma 2.1], in its turn, being that of [4, Corollary 2.3], we have:

Corollary 3.6. *Any abstract hereditary local subclass of a locally-finite quasi-variety is ultra-multiplicative.*

Lemma 3.7. *Let $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ be a [quasi-]variety. Then, $(\mathrm{Si} \mid \mathrm{SI}^\omega_{[\mathsf{Q}]})(\mathsf{Q}) \cup \mathbf{I}(\prod \varnothing))$ is local.*

Proof. Consider any $\mathfrak{B} \in (\mathsf{Q} \setminus ((\mathrm{Si} \mid \mathrm{SI}^\omega_{[\mathsf{Q}]})(\mathsf{Q}) \cup \mathbf{I}(\prod \varnothing)))$, in which case there are some $\bar{a} \in (B^2 \setminus \Delta_B) \neq \varnothing$, $n \in (\omega|\{1\})$ and $\bar{\theta} \in (\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus (\mathrm{img}\, \bar{\vartheta}^B))^n$, where, for any $C \subseteq B$, $\bar{\vartheta}^C \triangleq (\langle \Delta_C \rangle | \langle \Delta_C, C^2 \rangle)$, |"such that $(B^2 \cap (\bigcap (\mathrm{img}\, \bar{\theta}))) = \Delta_B$", and so some $\langle \bar{b}^{i,j} \rangle^{j \in (1|2)}_{i \in n} \in (\prod^{j \in (1|2)}_{i \in n} ((\theta_i \setminus \vartheta^B_j) \cup (\vartheta^B_j \setminus \theta_i))) \neq \varnothing$. Let $\mathfrak{A}$ be the finitely-generated subalgebra of $\mathfrak{B}$ generated by $\{a_0, a_1\} \cup \{b^{i,j}_k \mid i \in n, j \in (1|2), k \in 2\}$, in which case, by Lemma 2.3 with $h = \Delta_A$, $\bar{\eta} \triangleq \langle \theta_i \cap A^2 \rangle_{i \in n} \in (\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{A}) \setminus (\mathrm{img}\, \bar{\vartheta}^A))^n$, as $\langle \bar{b}^{i,j} \rangle^{j \in (1|2)}_{i \in n} \in (\prod^{j \in (1|2)}_{i \in n} ((\eta_i \setminus \vartheta^A_j) \cup (\vartheta^A_j \setminus \eta_i)))$, and so $\mathfrak{A} \in (\mathsf{Q} \setminus ((\mathrm{Si} \mid \mathrm{SI}^\omega_{[\mathsf{Q}]})(\mathsf{Q}) \cup \mathbf{I}(\prod \varnothing)))$, for $\bar{a} \in (A^2 \setminus \Delta_A)$ |"and $(A^2 \cap (\bigcap (\mathrm{img}\, \bar{\eta}))) = (A^2 \cap \Delta_B) = \Delta_A$". □

This immediately yields:

Corollary 3.8. *Any locally-finite [relatively] semi-simple [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ with hereditary $\mathrm{SI}^\omega_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{I}(\prod \varnothing))$ is [relatively] finitely semi-simple.*

4. Main issues

A *[restricted]* $\{$*quaternary*$\}$ Σ-*(equational)system/scheme of rank* $\alpha \in ((\infty[\cap 5]) \setminus 4)$ is any $\mho \subseteq \mathrm{Eq}^\alpha_\Sigma$ defining the [quantifier-free] formula $\Phi^\alpha_\mho \triangleq (\exists_{\beta \in (\alpha \setminus 4)} x_\beta (\bigwedge \mho))$ with free variables in V_4, in which case, for all $\mathfrak{A}, \mathfrak{B} \in \mathsf{A}_\Sigma$, $\bar{a} \in A^4$ and $h \in \hom(\mathfrak{A}, \mathfrak{B})$:

$$(\mathfrak{A} \models \Phi^\alpha_\mho[x_i/a_i]_{i \in 4}) \Rightarrow (\mathfrak{B} \models \Phi^\alpha_\mho[x_i/h(a_i)]_{i \in 4}), \tag{4.1}$$

and so, for any set I with [any sub]direct product $\mathfrak{D}$ of any $\overline{\mathfrak{C}} \in \mathsf{A}^I_\Sigma$:

$$(\mathfrak{D} \models \Phi^\alpha_\mho[x_i/\bar{a}^j]_{j \in 4}) \Leftrightarrow (\forall i \in 4 : \mathfrak{C}_i \models \Phi^\alpha_\mho[x_j/a^j_i]_{j \in 4}) \tag{4.2}$$

for all $\langle \bar{a}^j \rangle_{j \in 4} \in D^4$, as $\forall i \in I : (\pi_i \restriction D) \in \hom(\mathfrak{D}, \mathfrak{C}_i)$. Then, $\mho$ is called an *implication system/scheme for* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, if this satisfies the Σ-implication:

$$(\{x_0 \approx x_1\} \cup \mho) \to (x_2 \approx x_3). \tag{4.3}$$

[Likewise, it is called an *identity*|*reflexive*|*symmetric*|*transitive* one, if K satisfies the Σ-implications of the form $(\varnothing|\varnothing|\mho|(\mho \cup (\mho[x_{2+i}/x_{3+i}]_{i \in 2}))) \to \Psi$, where

$\Psi \in (\mho([x_3/x_2]|[x_{2+i}/x_i]_{i\in 2}|[x_3/x_2, x_2/x_3]|[x_3/x_4]))$, reflexive symmetric transitive ones being called *equivalence* ones. Then, $\mho$ is called a *congruence* one, if it is an equivalence one, while, for each $\varsigma \in \Sigma$ of arity $n \in (\omega \setminus 1)$, K satisfies the Σ-implications of the form $(\bigcup_{j\in n}(\mho[x_{2+i}/x_{2+i+(2\cdot j)}]_{i\in 2})) \to \Psi$, where $\Psi \in (\mho[x_{2+i}/\varsigma(\langle x_{2+i+(2\cdot j)}\rangle_{j\in n})]_{i\in 2})$.] Given "a {quasi-}variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ {⟨not necessarily⟩} equal to"/ a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, a finite|[finite] Σ-scheme $\mho$ of rank $m \in ((\omega[\cap\{4\}])|\{4\})$ is called a *[restricted]| "equationally definable principal {⟨Q−⟩relative} congruences ({⟨Q−⟩}[R/EDP{R}C)"/implicative|disjunctive scheme/system for* K, if $\forall\mathfrak{A} \in \mathsf{K}, \forall\bar{a} \in A^4 : (\forall\theta \in (\mathrm{Co}_{\{\mathsf{Q}\}}(\mathfrak{A})/\{\Delta_A\}) : (\langle a_0, a_1\rangle \in | \notin \theta) \Rightarrow (\langle a_2, a_3\rangle \in \theta)) \Leftrightarrow (\mathfrak{A} \models \Phi^m_\mho[x_i/a_i]_{i\in 4})$ (cf. [5]/[18]|[17]) /"so for $\mathbf{I}(\{\prod\varnothing\} \cup (([\mathbf{S}]\mathbf{P}^{\mathrm{U}})|(\mathbf{S}[\mathbf{P}^{\mathrm{U}}]))\mathsf{K})$" "being an [identity] implication scheme for $\mathsf{K} \cap ((\mathsf{A}_\Sigma\{\cap\mathsf{Q}\})/\mathsf{A}_\Sigma)$"|, ⌈/"⌊quasi-⌋varieties|pre-varieties of"⌉‖"the class of" Σ-algebras ⌈/"with ⌊relatively⌋ subdirectly-irreducibles"|"generated by subclasses"⌉‖"in a $\mathsf{C} \subseteq \mathsf{A}_\Sigma$" with /⌈[restricted|finite]⌉‖ "{Q-}[R]EDP{R}C scheme"/"implicative|disjunctive system" $\mho$ being "/called"‖denoted /"⌈*[restricted|finitely]*⌉ $\mho$*-implicative|-disjunctive*"‖"by $\mathsf{C}^{\mho/}_{\{\mathsf{Q}\}/\mho}$".

4.1. **Disjunctive pre-varieties.** Given any restricted Σ-equational system $\mho$ and any $\mathfrak{A} \in \mathsf{A}_\Sigma$, we have $\mho^{\mathfrak{A}} : (A^2)^2 \to \wp(A^2), \langle\bar{a}, \bar{b}\rangle \mapsto \{\langle\phi^{\mathfrak{A}}[x_i/a_i, x_{2+i}/b_i]_{i\in 2}, \psi^{\mathfrak{A}}[x_i/a_i, x_{2+i}/b_i]_{i\in 2}\rangle \mid (\phi \approx \psi) \in \mho\}$. Then, any $\theta \in (\mathrm{Co}(\mathfrak{A})[\cap\{\Delta_A\}])$ is $\mho^{\mathfrak{A}}$-disjunctive iff $\mathfrak{A}/\theta$ [viz., $\mathfrak{A} \in \mathbf{I}(\mathfrak{A}/\theta)$, for $\nu_\theta \in \hom(\mathfrak{A}, \mathfrak{A}/\theta)$ is injective] is $\mho$-disjunctive, in which case, for any $\vartheta \in (\mathrm{Co}(\mathfrak{A}) \cap \wp(\theta, A^2))[= \mathrm{Co}(\mathfrak{A})]$, the Σ-identities in $\bigcup_{j\in 2}(\mho[x_{(2\cdot j)+1}/x_{2\cdot j}])$, being true in $\mathfrak{A}/\theta$, are so in $(\mathfrak{A}/\vartheta) \in \mathbf{H}(\mathfrak{A}/\theta)$, as, by the Homomorphism Theorem, $(\nu_\vartheta \circ \nu_\theta^{-1}) \in \hom(\mathfrak{A}/\theta, \mathfrak{A}/\vartheta)$, i.e., ϑ is weakly $\mho^{\mathfrak{A}}$-disjunctive [and so is $\mathrm{Cg}^{\mathfrak{A}}$]. In this way, by Lemmas 2.3, 3.1, 3.2 and Remark 2.2, we, first, get:

Lemma 4.1. *Let* $\mho \subseteq \mathrm{Eq}^4_\Sigma$. *Then, any* $\mho$*-disjunctive [pre-]variety* $\mathsf{P} \subseteq \mathsf{A}_\Sigma$, *being generated by* $\mathsf{P}_\mho \subseteq \mathbf{IS}\mathsf{P}_\mho \subseteq \mathsf{P}_\mho$, *is equal to* $\mathbf{IP}^{\mathrm{SD}}\mathsf{P}_\mho$, *in which case it is [relatively both] congruence-distributive [and finitely-subdirectly-representable] with* $\mathsf{P}_\mho = (\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathbf{I}(\prod\varnothing))$, *and so* $\mathrm{SI}^\omega_{[(\mathsf{P})]}(\mathsf{P}) = (\mathsf{P}_\mho \setminus \mathbf{I}(\prod\varnothing))$. *In particular, any /finite* $\mho$*-disjunctive* $\mathfrak{A} \in \mathsf{A}_\Sigma$ *is finitely/ subdirectly-irreducible.*

Theorem 4.2. *Any [pre-]variety* $\mathsf{P} \subseteq \mathsf{A}_\Sigma$ *is disjunctive iff it is [relatively both] congruence-distributive [and finitely-subdirectly-representable] with* $\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathbf{I}(\prod\varnothing)$ *being "a universal (infinitary) model class"/hereditary.*

Proof. The "only if" part is by Lemma 4.1. Conversely, assume P is [relatively both] congruence-distributive [and finitely-subdirectly-representable] with hereditary $\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathbf{I}(\prod\varnothing)$, in which case, by Corollary 2.5, it is [relatively] finitely-subdirectly-representable, while, by Lemma 2.3, $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma) \cap \wp(\theta, \mathrm{Eq}^4_\Sigma)$, where $\theta \triangleq (\mathrm{Eq}^4_\Sigma \cap (\bigcap \mathrm{Co}_{\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})}(\mathfrak{Tm}^4_\Sigma))) \in \mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma)$, is distributive, for $\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma/\theta)$ is so. Let $\forall j \in 2 : \vartheta_j \triangleq \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\theta \cup \{\langle x_{2\cdot j}, x_{(2\cdot j)+1}\rangle\}) \in (\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma) \cap \wp(\theta, \mathrm{Eq}^4_\Sigma)) \ni \mho \triangleq (\vartheta_0 \cap \vartheta_1) \subseteq \mathrm{Eq}^4_\Sigma$. Consider any $\mathfrak{A} \in \mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})$ and any $\bar{a} \in A^4$. Let $h \in \hom(\mathfrak{Tm}^4_\Sigma, \mathfrak{A})$ extend $\{\langle x_i, a_i\rangle \mid i \in 4\}$, in which case $\mathfrak{B} \triangleq (\mathfrak{A}\restriction(\mathrm{img}\, h)) \in (\mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P}) \cup \mathbf{I}(\prod\varnothing))$, and so $((\{\langle a_0, a_1\rangle, \langle a_2, a_3\rangle\} \cap \Delta_A) \neq \varnothing) \;\&|\Leftrightarrow (\mathfrak{A} \models \Phi^4_\mho[h\restriction V_4])$, unless $\mathfrak{B} \in \mathrm{SI}^\omega_{[\mathsf{P}]}(\mathsf{P})$. Otherwise, by Lemma 2.3, Corollary 2.4 and the Homomorphism Theorem, $\theta \subseteq \eta \triangleq (\ker h) \in \mathrm{MI}^\omega(\mathrm{Co}_{[\mathsf{P}]}(\mathfrak{Tm}^4_\Sigma))$, in which case we have:

$$\begin{gathered}(\mathfrak{A} \models \Phi^4_\mho[h\restriction V_4]) \Leftrightarrow ((\vartheta_0 \cap \vartheta_1) = \mho \subseteq \eta) \Leftrightarrow (\eta = \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup (\vartheta_0 \cap \vartheta_1)) = \\ (\mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup \vartheta_0) \cap \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup \vartheta_1)) \Leftrightarrow (\exists j \in 2 : \eta = \mathrm{Cg}^{\mathfrak{Tm}^4_\Sigma}_{[\mathsf{P}]}(\eta \cup \vartheta_j)) \Leftrightarrow \\ (\exists j \in 2 : \vartheta_j \subseteq \eta) \Leftrightarrow (\exists j \in 2 : \langle x_{2\cdot j}, x_{(2\cdot j)+1}\rangle \in \eta) \Leftrightarrow (\exists j \in 2 : a_{2\cdot j} = a_{(2\cdot j)+1}),\end{gathered}$$

and so $\mho$ is a disjunctive system for $\mathrm{SI}^{\omega}_{[\mathsf{P}]}(\mathsf{P})$. Thus, P, being [relatively] finitely-subdirectly-representable, is $\mho$-disjunctive, as required. $\square$

This, by Corollary 2.5 and Lemma 4.1 (as well as the Compactness Theorem for ultra-multiplicative classes; cf., e.g., [11]), immediately yields:

Corollary 4.3. *Any [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ is (finitely) disjunctive iff it is [relatively] congruence-distributive with $\mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{I}(\prod \varnothing)$ being "a universal (first-order) model class"/"hereditary (and ultra-multiplicative)".*

This, in its turn, by Corollaries 2.4, 3.6 and Lemma 3.7, immediately yields:

Corollary 4.4. *Any locally-finite [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ is (finitely) disjunctive iff it is [relatively] congruence-distributive with $\mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{I}(\prod \varnothing)$ being "a universal model class"/hereditary.*

Finally, this, by the congruence-distributivity of lattice expansions (cf. [12]) and Corollary 2.8, immediately yields:

Corollary 4.5. *Suppose $\Sigma_+ \subseteq \Sigma$. Then, any finitely-generated variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ of lattice expansions with hereditary $\mathrm{SI}^{(\omega)}(\mathsf{V}) \cup \mathbf{I}(\prod \varnothing)$ is finitely disjunctive.*

This provides an immediate (though far from being constructive) insight into the finite disjunctivity of the finitely-generated variety of distributive/Stone|"De Morgan" lattices/algebras|algebras||lattices, a constructive one being given by [16, Example 1/2] and [17, Lemma 11].

4.2. Implication systems versus EDPC.

Lemma 4.6. *Let $\mho$ be a finite Σ-system of rank $m \in (\omega \setminus 4)$, $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ a quasi-variety, $\mathfrak{A} \in \mathsf{A}_\Sigma$ and $\bar{a}, \bar{b} \in A^2$. Suppose $\mho$ is an implication system for Q such that $\mathfrak{A} \models \Phi^m_\mho[x_i/a_i, x_{2+i}/b_i]_{i \in 2}$. Then, $\bar{b} \in \theta \triangleq \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{Q}}(\{\bar{a}\})$.*

Proof. As, by (4.1), we have $(\mathfrak{A}/\theta) \models \Phi^m_\mho[x_i/\nu_\theta(a_i), x_{2+i}/\nu_\theta(b_i)]_{i \in 2}$, for $\nu_\theta \in \hom(\mathfrak{A}, \mathfrak{A}/\theta)$, while $\mathsf{Q} \ni (\mathfrak{A}/\theta)$ satisfies (4.3), and $\bar{a} \in \theta = (\ker \nu_\theta)$, we get $\bar{a} \in \theta$. $\square$

This, by (4.1) {and (4.2)} as well as Corollary 2.6, immediately yields:

Corollary 4.7. *Let $\mho$ be a finite [restricted] Σ-equational system and $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ a (quasi-)variety. {Suppose $\mho$ is an implication system for Q.} Then, $\mathsf{Q}^{\mho}_{(\mathsf{Q})}$ is abstract {and [sub-]multiplicative}.*

Theorem 4.8. *Any $\mho \in \wp_\omega(\mathrm{Eq}^4_\Sigma)$ is an REDPC scheme for a variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ iff it is an identity congruence implication one.*

Proof. The "only if" part is immediate. Conversely, if $\mho$ is an identity congruence implication scheme for V, then, by induction on construction of any $\varphi \in \mathrm{Tm}^\omega_\Sigma$, we conclude that V satisfies the Σ-identities in $\mho[x_{2+i}/(\varphi[x_0/x_i])]_{i \in 2}$, in which case, by Mal'cev Lemma [10] (cf. [5, Lemma 2.1]), for any $\mathfrak{A} \in \mathsf{V}$, $\bar{a} \in A^2$ and $\bar{b} \in \mathrm{Cg}^{\mathfrak{A}}(\{\bar{a}\})$, we have $\mathfrak{A} \models \Phi^4_\mho[x_i/a_i, x_{2+i}/b_i]_{i \in 2}$, and so Lemma 4.6 completes the argument. $\square$

4.3. Implicative quasi-varieties.

Lemma 4.9. *Let $\mho$ be a finite Σ-equational system of rank $m \in (\omega \setminus 4)$ and Q a [quasi-]variety (with implication system $\mho$). Then, $(\mathsf{Q}_\mho =)((\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{I}(\prod \varnothing)) \cap \mathsf{Q}_\mho) = ((\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{I}(\prod \varnothing)) \cap \mathsf{Q}^{\mho}_{[\mathsf{Q}]})$. In particular, Q is [relatively] semi-simple with $\mathsf{Q}_\mho = (\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{I}(\prod \varnothing))$, whenever it is $\mho$-implicative.*

Proof. Consider any $\mathfrak{A} \in \mathsf{Q}$. If it is [$\mathsf{Q}$-]simple|one-element, $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{A}) \subseteq \{\Delta_A, A^2\}$, implying $\forall \bar{a} \in A^4 : (\forall \theta \in \mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{A}) : (a_0 \; \theta \; a_1) \Rightarrow (a_2 \; \theta \; a_3)) \Leftrightarrow ((a_0 = a_1) \Rightarrow (a_2 = a_3))$, as $\langle a_2, a_3 \rangle \in A^2$, so $(\mathfrak{A} \in \mathsf{Q}_{\mho}) \Leftrightarrow (\mathfrak{A} \in \mathsf{Q}^{\mho}_{[\mathsf{Q}]})$. (Now, assume $\mathfrak{A}$ is $\mho$-implicative. Consider any $\theta \in (\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{A}) \setminus \{\Delta_A\})$ and take any $\bar{a} \in (\theta \setminus \Delta_A) \neq \varnothing$, implying $\forall a_2, a_3 \in A : \mathfrak{A} \models \Phi^m_{\mho}[x_i/a_i]_{i\in 4}$, so, as (4.3) is true in each member of $\mathsf{Q} \ni (\mathfrak{A}/\theta) \models \Phi^m_{\mho}[x_i/\nu_\theta(a_i)]_{i\in 4}$, by (4.1), since $\nu_\theta \in \hom(\mathfrak{A}, \mathfrak{A}/\theta)$, $\langle a_2, a_3 \rangle$ is in $\theta = (\ker \nu_\theta)$, for $\bar{a}$ is so. Thus, $\theta = A^2$.) So, Corollary 2.5 ends the proof. □

This, by Corollaries 2.5 and 4.7, immediately yields:

Corollary 4.10. *Let $\mho$ be a (restricted) Σ-system. Then, a [quasi-]variety is $\mho$-implicative if(f) it is [relatively] semi-simple with (R)EDP[R]C scheme $\mho$.*

Corollary 4.11. *Let $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ be a [quasi-]variety and $\mho \in \wp_\omega(\mathrm{Eq}^4_\Sigma)$. Then, Q is restricted $\mho$-implicative iff it is the quasi-/pre-variety generated by a subclass with restricted implicative system $\mho$, in which case it is finitely disjunctive, and so [relatively] both congruence-distributive and finitely-semi-simple.*

Proof. The "only-if" part is by Corollary 2.5. Conversely, assume $\mathsf{Q} = \mathbf{ISPP}^{\mathrm{U}}\mathsf{K}$, for some $\mathsf{K} \subseteq \mathsf{Q}_{\mho}$, in which case $\mho$, being finite, is an implication system for Q, while $\mathsf{K}' \triangleq \mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathbf{P}^{\mathrm{U}}\mathsf{Q}_{\mho} \subseteq \mathsf{Q}_{\mho}$, and so Q is the pre-variety generated by $\mathsf{K}' \subseteq \mathsf{Q}_{\mho}$. Then, by [18, Remark 2.4], K' has a finite disjunctive system, in which case Q is finitely disjunctive, and so, by Lemma 4.1, is [relatively] congruence-distributive. Hence, by Corollary 2.8 and Lemma 4.9, $\mathrm{SI}^{(\omega)}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq (\mathbf{ISP}^{\mathrm{U}}\mathsf{K}' \setminus \mathbf{I}(\prod \varnothing)) \subseteq (\mathbf{ISP}^{\mathrm{U}}\mathsf{Q}_{\mho} \setminus \mathbf{I}(\prod \varnothing)) \subseteq (\mathsf{Q}_{\mho} \setminus \mathbf{I}(\prod \varnothing)) \subseteq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$, as required. □

This, in particular, means that the notion of restricted implicative quasi-variety adopted here is equivalent to that of implicative quasi-variety adopted in [18].

Lemma 4.12. *Let $\mho$ be a finite [restricted] Σ-equational system of rank $m \in \omega$, $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ a [restricted] $\mho$-implicative quasi-variety and $I \in \Upsilon$ with [a sub]direct product $\mathfrak{B}$ of an $\overline{\mathfrak{A}} \in \mathrm{SI}_{\mathsf{Q}}(\mathsf{Q})^I$. Then, for any $\Theta \in \wp_\omega(B^2)$, $\mathrm{Cg}^{\mathfrak{B}}_{\mathsf{Q}}(\Theta) = \theta^B_{\mathcal{F}_\Theta}$, where $\mathcal{F}_\Theta \triangleq \wp(I_\Theta, I)$ and $I_\Theta \triangleq (I \cap (\bigcap_{i\in I} \varepsilon_I^{-1}[\Theta])$.*

Proof. By induction on $|\Theta|$. If $\Theta = \varnothing$, then $I_\Theta = I$, in which case $\mathcal{F}_\Theta = \{I\}$, and so $\mathrm{Cg}^{\mathfrak{B}}_{\mathsf{Q}}(\Theta) = \Delta_B = \theta^B_{\mathcal{F}_\Theta}$. Otherwise, take any $\langle \bar{a}, \bar{b} \rangle \in \Theta$, in which case $\Xi \triangleq (\Theta \setminus \{\langle \bar{a}, \bar{b} \rangle\}) \in \wp_\omega(B^2)$ with $|\Xi| < |\Theta|$, and so, by induction hypothesis, $\theta \triangleq \mathrm{Cg}^{\mathfrak{B}}_{\mathsf{P}}(\Xi) = \theta^B_{\mathcal{F}_\Xi} = (B^2 \cap (\bigcap_{i\in I_\Xi} \ker \pi_i)) = (B^2 \cap (\bigcap_{i\in I} (\pi_i)_2^{-1}[\eta_i])$ with $\bar{\eta} \in (\prod_{i\in I} \mathrm{Co}_{\mathsf{Q}}(\mathfrak{A}_i))$ given by $\forall i \in (I_\Xi|(I \setminus I_\Xi)) : \eta_i \triangleq (\Delta_{A_i}|A_i^2)$. Then, for each $i \in I$, $g_i \triangleq ((\pi_i \restriction B) \circ \nu_{\eta_i}) \in \hom(\mathfrak{B}, \mathfrak{C}_i)$ with $\mathfrak{C}_i \triangleq (\mathfrak{A}/\eta_i) \in \mathsf{Q}^{\mho}_{\mathsf{Q}}$, in view of Lemma 4.9, in which case, by Lemma 2.3, $\vartheta_i \triangleq (\ker g_i) = (B^2 \cap (\pi_i)_2^{-1}[\eta_i]) \in \mathrm{Co}(\mathfrak{B})$, and so $\theta = (B^2 \cap (\bigcap_{i\in I} \vartheta_i))$. Therefore, $\theta \subseteq \vartheta_i$, in which case, by the Homomorphism Theorem, $f_i \triangleq (\nu_\theta^{-1} \circ g_i) \in \hom(\mathfrak{B}/\theta, \mathfrak{C}_i)$ with $g_i = (\nu_\theta \circ f_i)$, and so $((B/\theta) \cap (\bigcap_{i\in I} (\ker f_i))) = \Delta_{B/\theta}$. Hence, $e : (B/\theta) \to (\prod_{i\in I} C_i), b \mapsto \langle f_i(b) \rangle_{i\in I}$ is an embedding of $\mathfrak{B}/\theta$ into $\mathfrak{C} \triangleq (\prod_{i\in I} \mathfrak{C}_i)$ with $\forall i \in I : f_i = (e \circ \pi_i)$, and so an isomorphism from $\mathfrak{B}/\theta$ onto $\mathfrak{D} \triangleq (\mathfrak{C} \restriction e[B/\theta])$ with

$$\forall i \in I : g_i = (h \circ \pi_i), \tag{4.4}$$

where $h \triangleq (\nu_\theta \circ e) \in \hom(\mathfrak{B}, \mathfrak{D})$ with $h[B] = D$ and $(\ker h) = (\ker \nu_\theta) = \theta$, for e is injective. Next, for every $i \in I$, $\pi_i[B] = A_i$, in which case, by (4.4), $\pi_i[D] = \pi_i[h[B]] = g_i[B] = \nu_{\eta_i}[\pi_i[B]] = C_i$, and so $\mathfrak{D}$ is a subdirect product of $\overline{\mathfrak{C}} \triangleq \langle \mathfrak{C}_i \rangle_{i\in I}$. Moreover, if $B = (\prod_{i\in I} A_i)$, then, for each $\bar{c} \in E$ and every $i \in I$, there is some $a_i \in A_i$ with $\nu_{\eta_i}(a_i) = c_i$, in which case $\bar{a} \triangleq \langle a_i \rangle_{i\in I} \in B$ with $D = h[B] \ni h(\bar{a}) = \bar{c}$, in view of (4.4), and so $D = E$. Then, by

Corollaries 2.5 and 4.7, $\mathfrak{D} \in \mathsf{Q}^{\mho}_{\mathsf{Q}} \supseteq (\mathrm{img}\,\overline{\mathfrak{C}})$, in which case, as, by (4.2, 4.4), for all $\bar{c}, \bar{d} \in D$, we have $(\mathfrak{D} \models \Phi^m_{\mho}[x_0/h(\bar{a}), x_1/h(\bar{b}), x_2/\bar{c}, x_3/\bar{d}]) \Leftrightarrow (\forall i \in I : \mathfrak{C}_i \models \Phi^m_{\mho}[x_0/\nu_{\eta_i}(a_i), x_1/\nu_{\eta_i}(b_i), x_2/c_i, x_3/d_i])$, we get $\mathrm{Cg}^{\mathfrak{D}}_{\mathsf{Q}}(\{\langle h(\bar{a}), h(\bar{b})\rangle\}) = (D^2 \cap (\bigcap_{i\in I} (\pi_i)_2^{-1}[\mathrm{Cg}^{\mathfrak{C}_i}_{\mathsf{Q}}(\langle \nu_{\eta_i}(a_i), \nu_{\eta_i}(b_i)\rangle)]))$, and so, applying Corollary 2.6 twice and taking (4.4) into account, we eventually get:

$$\begin{aligned}\mathrm{Cg}^{\mathfrak{B}}_{\mathsf{Q}}(\Theta) &= \mathrm{Cg}^{\mathfrak{B}}_{\mathsf{Q}}((\ker h) \cup \{\langle \bar{a}, \bar{b}\rangle\}) = h_2^{-1}[\mathrm{Cg}^{\mathfrak{D}}_{\mathsf{Q}}(\{\langle h(\bar{a}), h(\bar{b})\rangle\})] = \\ &(B^2 \cap (\bigcap\nolimits_{i\in I} h_2^{-1}[(\pi_i)_2^{-1}[\mathrm{Cg}^{\mathfrak{C}_i}_{\mathsf{Q}}(\langle \nu_{\eta_i}(a_i), \nu_{\eta_i}(b_i)\rangle)]])) = \\ &(B^2 \cap (\bigcap\nolimits_{i\in I} (\pi_i)_2^{-1}[(\nu_{\eta_i})_2^{-1}[\mathrm{Cg}^{\mathfrak{C}_i}_{\mathsf{Q}}(\langle \nu_{\eta_i}(a_i), \nu_{\eta_i}(b_i)\rangle)]])) = \\ &(B^2 \cap (\bigcap\nolimits_{i\in I} (\pi_i)_2^{-1}[\mathrm{Cg}^{\mathfrak{A}_i}_{\mathsf{Q}}(\eta_i \cup \mathrm{Cg}^{\mathfrak{A}_i}_{\mathsf{Q}}(\{\langle a_i, b_i\rangle\}))])) = \\ &(B^2 \cap (\bigcap\nolimits_{i\in I_\Theta} \ker \pi_i)) = \theta^B_{\mathcal{F}_\Theta},\end{aligned}$$

because, for all $i \in I$, $(\mathrm{Cg}^{\mathfrak{A}_i}_{\mathsf{Q}}(\{\langle a_i, b_i\rangle\}) = (\Delta_{A_i}|A_i^2)) \Leftrightarrow (a_i = | \neq b_i)$, since, by Lemma 4.9, $\mathfrak{A}_i \in \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q})$. □

Corollary 4.13. *Let $\mho$ be a finite [restricted] Σ-equational system, $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ a [restricted] $\mho$-implicative quasi-variety and $I \in \Upsilon$ with [a sub]direct product $\mathfrak{B}$ of an $\overline{\mathfrak{A}} \in \mathrm{SI}_{\mathsf{Q}}(\mathsf{Q})^I$. Then, for any $\Xi \subseteq B^2$, $\mathrm{Cg}^{\mathfrak{B}}_{\mathsf{Q}}(\Xi) = \theta^B_{\mathcal{F}_\Xi}$, where $\mathcal{F}_\Xi \triangleq \mathrm{Fg}_I(\varepsilon_I[\Xi])$. In particular, Q is relatively [sub]directly filtral.*

Proof. As $\mathrm{Cg}^{\mathfrak{B}}_{\mathsf{Q}}$ is finitary, $(\mathrm{Cg}^{\mathfrak{B}}_{\mathsf{Q}}(\Xi) = (\bigcup \mathrm{Cg}^{\mathfrak{B}}_{\mathsf{Q}}[\wp_\omega(\Xi)])$. Likewise, as Fg_I is finitary, $\mathrm{Fg}_I(\varepsilon_I[\Xi]) = (\bigcup \mathrm{Fg}_I[\wp_\omega(\varepsilon_I[\Xi])]) = (\bigcup\{\mathrm{Fg}_I(\varepsilon_I[\Theta]) \mid \Theta \in \wp_\omega(\Xi)\}) = (\bigcup\{\wp(I_\Theta, I) \mid \Theta \in \wp_\omega(\Xi)\})$. In this way, (2.5) and Lemma 4.12 complete the argument. □

Lemma 4.14. *Let $\alpha' \in (\infty\setminus\omega)$ and Q a relatively (sub)directly filtral quasi-variety. Then, $\mathrm{SI}_{\mathsf{Q}}(\mathsf{Q})_{(\alpha'+1)(\cup\infty)}$ has a (restricted) implicative system.*

Proof. Let $\alpha \triangleq (\alpha'(\cap 4))$, $\forall i \in I \triangleq \{\theta \in \mathrm{Co}_{\mathrm{SI}_{\mathsf{Q}}(\mathsf{Q})}(\mathfrak{Tm}^\alpha_\Sigma) \mid (x_0\ \theta\ x_1) \Rightarrow (x_2\ \theta\ x_3)\} : \mathfrak{B}_i \triangleq (\mathfrak{Tm}^\alpha_\Sigma/i) \in \mathrm{SI}_{\mathsf{Q}}(\mathsf{Q})$, $\forall j \in 4 : a_j \triangleq \langle \nu_i(x_j)\rangle_{i\in I}$, $\mathfrak{C}$ the subalgebra of $\mathfrak{F} \triangleq (\prod_{i\in I} \mathfrak{B}_i)$ generated by $A_4 \triangleq \{a_j \mid j \in 4\}$ and $\mathfrak{D} \triangleq (\mathfrak{F}{\restriction}(F(\cap E)))$, in which case (for each $i \in I$, as $\mathfrak{B}_i$ is generated by $\nu_i[V_4] = \pi_i[A_4]$, we have $\pi_i[D] = B_i$, and so) $\exists \mathcal{F} \in \mathrm{Fi}(I) : \mathrm{Co}_{\mathsf{Q}}(\mathfrak{D}) \ni \vartheta \triangleq \mathrm{Cg}^{\mathfrak{D}}_{\mathsf{Q}}(\{\langle a_0, a_1\rangle\}) = \theta^D_{\mathcal{F}}$. Then, as $\langle a_0, a_1\rangle \in \vartheta$, i.e., $\varepsilon_I(a_2, a_3) \supseteq \varepsilon_I(a_0, a_1) \in \mathcal{F}$, we get $\varepsilon_I(a_2, a_3) \in \mathcal{F}$, that is, $\langle a_2, a_3\rangle \in \vartheta$. Let $\gamma \triangleq (\omega \cup |F\setminus E|) \supseteq \omega$ and $\beta \triangleq (\gamma(\cap 4))$, in which case $|\gamma\setminus 4| = |\gamma| \supseteq |F\setminus E|$, and so there is a surjection from $V_\beta\setminus V_4$ onto $D\setminus E$ to be extended to a surjective $h \in \hom(\mathfrak{Tm}^\beta_\Sigma, \mathfrak{D})$ including $\{\langle x_j, a_j\rangle \mid j \in 4\}$. Consider any $\mathfrak{A} \in \mathsf{Q}$ and any $g \in \hom(\mathfrak{Tm}^\beta_\Sigma, \mathfrak{A})$ with $(\ker h) \subseteq (\ker g) \ni \langle x_0, x_1\rangle$, in which case, by the Homomorphism Theorem, $f \triangleq (h^{-1} \circ g) \in \hom(\mathfrak{D}, \mathfrak{A})$, and so, by Lemma 2.3, $\langle a_0, a_1\rangle \in (\ker f) = f_2^{-1}[\Delta_A] \in \mathrm{Co}_{\mathsf{Q}}(\mathfrak{D})$, for $g = (h \circ f)$ and $\Delta_A \in \mathrm{Co}_{\mathsf{Q}}(\mathfrak{A})$, as $\nu_{\Delta_A} \in \hom(\mathfrak{A}, \mathfrak{A}/\Delta_A)$ is bijective. Then, $\langle a_2, a_3\rangle \in \vartheta \subseteq (\ker f)$, in which case $g(x_2) = f(h(x_2)) = f(a_2) = f(a_3) = f(h(x_3)) = g(x_3)$, and so $\Omega \triangleq (\ker h)$ is an implication system for Q of rank β such that $\mathfrak{D} \models (\bigwedge \Omega)[h{\restriction}V_\beta]$, for $\Omega \subseteq (\ker h)$. Hence, by the Compactness Theorem for ultra-multiplicative classes (cf., e.g. [11]), some $\Xi \in \wp_\omega(\Omega)$ is an implication system for Q of rank β such that $\mathfrak{D} \models (\bigwedge \Xi)[h{\restriction}V_\beta]$, for $\Xi \subseteq (\ker h)$, in which case $V \triangleq ((\bigcup \mathrm{Var}[\Xi]) \setminus V_4)$ is finite, and so there is a bijection σ from V onto $V_m \setminus V_4$, where $m \triangleq (|V| + 4) \in (\omega \setminus 4)$. Thus, $\mho \triangleq (\Xi[\sigma]) \in \wp_\omega(\mathrm{Eq}^m_\Sigma)$ is an implication system for Q such that $\mathfrak{D} \models (\bigwedge)[(\sigma^{-1} \cup \Delta_{V_4}) \circ h]$, so $\mathfrak{D} \models \Phi^m_{\mho}[x_n/a_n]_{n\in 4}$. Now, consider any $\mathfrak{C} \in \mathrm{SI}_{\mathsf{Q}}(\mathsf{Q})_{(\alpha'+1)(\cup\infty)}$ and any $\bar{c} \in C^4$, in which case (4.3) is true in $\mathfrak{C}$, and so $(c_0 = c_1) \Rightarrow (c_2 = c_3)$, whenever $\mathfrak{C} \models \Phi^m_{\mho}[x_i/c_i]_{i\in 4}$. Conversely, assume $(c_0 = c_1) \Rightarrow (c_2 = c_3)$. Let $\mathfrak{G}$ be the subalgebra of $\mathfrak{C}$ generated by $\{c_i \mid i \in 4\}$ and $\mathfrak{H} \triangleq (\mathfrak{C}{\restriction}(C(\cap G))) \in (\mathrm{SI}_{\mathsf{Q}}(\mathsf{Q})(\cup \mathbf{I}(\prod \varnothing)))$ (in view of Lemma 2.10, in which case $\mathfrak{C} \models \Phi^m_{\mho}[x_i/c_i]_{i\in 4}$, whenever $|H| = 1$. Otherwise, $\mathfrak{H} \in \mathrm{SI}_{\mathsf{Q}}(\mathsf{Q})$.) Then, since $|\alpha' \setminus 4| = |\alpha'| \supseteq |H \setminus G|$, there is a surjection from $V_\alpha \setminus V_4$ onto $H \setminus G$ to be extended to a surjective $e \in \hom(\mathfrak{Tm}^\alpha_\Sigma, \mathfrak{H})$ including $\{\langle x_j, c_j\rangle \mid j \in 4\}$, in which case, by the Homomorphism Theorem, $(\nu_\eta^{-1} \circ e) \in \hom(\mathfrak{Tm}^\alpha_\Sigma/\eta, \mathfrak{H})$, where $\eta \triangleq (\ker e)$, is bijective, and so, by Corollary 2.4, $\eta \in I$. Therefore, by (4.1), $\mathfrak{H} \models \Phi^m_{\mho}[x_i/c_i]_{i\in 4}$, for $\{\langle a_i, c_i\rangle \mid i \in 4\} \subseteq ((\pi_\eta \restriction D) \circ (\nu_\eta^{-1} \circ e)) \in \hom(\mathfrak{D}, \mathfrak{H})$, so $\mathfrak{C} \models \Phi^m_{\mho}[x_i/c_i]_{i\in 4}$. In this way, $\mho$ is a (restricted) implicative system for $\mathrm{SI}_{\mathsf{Q}}(\mathsf{Q})_{(\alpha'+1)(\cup\infty)}$, as required. $\Box$

Lemma 4.15. *Any class K of Σ-algebras has an implicative system, if, for each $\alpha \in (\infty \setminus \omega)$, $\mathsf{K}_{\alpha+1}$ does so.*

Proof. Suppose, for every $\mho \in E \triangleq \wp_\omega(\mathrm{Eq}^\omega_\Sigma)$, $\mathsf{K}'_\mho \triangleq (\mathsf{K} \setminus \mathsf{K}_\mho) \neq \varnothing$, in which case $\varnothing \neq C_\mho \triangleq \{|A| \mid \mathfrak{A} \in \mathsf{K}'_\mho\} \subseteq \infty$, and so $\alpha_\mho \triangleq (\bigcap C_\mho) \in C_\mho$. Then, $\alpha \triangleq (\bigcup(\{\omega\} \cup \{\alpha_\mho \mid \mho \in E\})) \in (\infty \setminus \omega)$, for E is a set, in which case, for each $\mho \in E$, $(C_\mho \cap (\alpha_\mho + 1)) \ni \alpha_\mho \subseteq \alpha$, and so $\varnothing \neq (\mathsf{K}'_\mho)_{\alpha_\mho+1} \subseteq \mathsf{K}_{\alpha+1}$, as required. $\Box$

Lemmas 4.14 and 4.15 immediately yield:

Corollary 4.16. *Any [relatively] (sub)directly filtral [quasi-]variety Q is (restricted) implicative.*

Lemma 4.17. *Let $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ be a [quasi-]variety, $I \in \Upsilon$, $\overline{\mathfrak{A}} \in \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})^I$, $\mathfrak{D} \triangleq (\prod \overline{\mathfrak{A}})$, $\mathfrak{B} \in (\mathbf{S})\{\mathfrak{D}\}$ and $\theta \in (\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\})$. Suppose $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})^I \cup \mathbf{I}(\prod \varnothing)$ is ultra-multiplicative (and hereditary) $\{$while $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ is distributive$\}$. Then, θ is maximal in $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\}$ if$\{f\}$ it is ultra-filtral. In particular, all elements of $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$ are filtral, whenever Q is [relatively] both semi-simple and (sub)directly congruence-distributive.*

Proof. First, assume $\theta = \theta^B_{\mathfrak{U}}$, for some ultra-filter $\mathfrak{U}$ on I, in which case $\mathfrak{C} \triangleq (\mathfrak{D}/\theta^D_{\mathfrak{U}}) \in \mathbf{P}^{\mathrm{U}}\,\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ (while $h \triangleq (\Delta_B \circ \nu_{\theta^D_{\mathfrak{U}}}) \in \hom(\mathfrak{B}, \mathfrak{C})$, whereas $(\ker h) = (\Delta_B)_2^{-1}[\theta^D_{\mathfrak{U}}] = \theta$), and so (by the Homomorphism Theorem and Corollary 2.4), as $\theta \neq B^2$, $(\mathfrak{B}/\theta) \in ((\mathbf{IS})\mathbf{P}^{\mathrm{U}}(\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{I}(\prod\varnothing)) \setminus \mathbf{I}(\prod\varnothing)) \subseteq ((\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{I}(\prod\varnothing)) \setminus \mathbf{I}(\prod\varnothing)) = \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$. Then, by Lemma 2.3, $\theta \in \max(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\})$. $\{$Conversely, assume $\theta \in \max(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\}) \subseteq \mathrm{MI}(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}))$, in which case, by Lemma 2.7, there is some ultra-filter $\mathfrak{U}$ on I such that, as $\theta \neq B^2$, $(\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{B^2\}) \ni \theta^B_{\mathfrak{U}} \subseteq \theta$, and so, by the "if" part, $\theta = \theta^B_{\mathfrak{U}}$.$\}$ Finally, the inductivity of $\mathrm{Co}_{[\mathsf{Q}]}(\mathfrak{B})$, Remark 2.1, Lemma 2.3 and (2.5) (as well as Corollary 2.5) complete the argument. $\Box$

Then, by Corollaries 3.5, 4.11, 4.10, 4.13, 4.16 and Lemma 4.17, we eventually get:

Theorem 4.18. *A [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ is (restricted) implicative iff it is [relatively] (sub)directly filtral (iff it is [relatively] filtral) iff it is [relatively] both ($\{$finitely-$\}$)semi-simple and (sub)directly distributive with $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})^I \cup \mathbf{I}(\prod\varnothing)$ being "a (universal) first-order model class"/"ultra-multiplicative (and hereditary)" if(f) it is [relatively] ($\{$finitely-$\}$)semi-simple with an (R)EDP[R]C scheme.*

This, by Corollaries 2.4, 3.6 and Lemma 3.7, immediately yields:

Corollary 4.19. *Any locally-finite [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ is restricted implicative iff it is [relatively] both congruence-distributive and (finitely-)semi-simple with $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{I}(\prod\varnothing)$ being "a universal model class"/hereditary.*

Likewise, by Corollaries 4.3, 4.11 and Theorem 4.18, we immediately get:

Corollary 4.20. *A [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ is restricted implicative iff it is finitely disjunctive and [relatively] finitely-semi-simple.*

This, in its turn, by Corollaries 3.8 and 4.4, immediately yields:

Corollary 4.21. *A locally-finite [quasi-]variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ is restricted implicative iff it is (finitely) disjunctive and [relatively] {finitely-}semi-simple.*

Finally, by the congruence-distributivity of lattice expansions (cf. [12]), Corollaries 2.4, 2.8, 4.19, 4.21 and Lemma 4.9, we immediately get:

Corollary 4.22. *Suppose $\Sigma_+ \subseteq \Sigma$. Then, any locally-finite variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ of lattice expansions is semi-simple ["and (finitely) disjunctive"|"with hereditary $(\mathrm{Si}\,|\,\mathrm{SI})(\mathsf{V}) \cup \mathbf{I}(\prod\varnothing)$"] if[f] it is [restricted] implicative.*

Corollary 4.23. *Suppose $\Sigma_+ \subseteq \Sigma$, $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ a finite set of finite lattice expansions without non-simple non-one-element subalgebras and V the variety generated by K. Then, V is restricted implicative with $(\mathrm{Si}\,|\,\mathrm{SI})(\mathsf{V}) = (\mathbf{IS}\mathsf{K} \setminus \mathbf{I}(\prod\varnothing))$.*

These provide an immediate /{though far from being constructive} insight into the not/restricted implicativity of (and so not/ REDPC for; cf. Corollary 4.10) the not/ semi-simple finitely-generated variety of Stone/distributive|"De Morgan" algebras/lattices|algebras||lattices /(cf. [6]|[19]||) /"a constructive one being given below". Before, note that the stipulations of "relative ([sub]direct) congruence-distributivity"/"{finite} disjunctivity"/"lattice expanding" cannot be omitted in the formulations of Corollaries 4.3, 4.4, 4.5, 4.19, 4.20, 4.21, 4.22, 4.23 as well as Theorems 4.2 and 4.18, in view of:

Example 4.24. Let $\Sigma = \{\wedge\}$ and SL the variety of semi-lattices, in which case, for any filter $F \neq A$ of any $\mathfrak{A} \in \mathsf{SL}$, χ^F_A is a surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{S}_2 \in \mathsf{SL}$ with $S_2 \triangleq 2$ and $\wedge^{\mathfrak{A}} \triangleq (\cap\restriction 2^2)$, and so, by Remark 2.2, $\mathsf{SL} = \mathbf{IP}^{\mathrm{SD}}\mathfrak{S}_2$. Now, assume $|A| > 2$, in which case, providing $\mathfrak{A}$ is a chain, for any $\bar{a} \in A^3$ with $|\operatorname{img}\bar{a}| = 3$ such that $a_0 \leqslant^{\mathfrak{A}} a_1 \leqslant^{\mathfrak{A}} a_2$ and $i \in 2$, $\Delta_A \neq \theta_i \triangleq ([a_i, a_{i+1}]^2_{\mathfrak{A}} \cup \Delta_A) = \mathrm{Cg}^{\mathfrak{A}}(\{\langle a_i, a_{i+1}\rangle\}) \in \mathrm{Co}(\mathfrak{A})$, while $(\theta_0 \cap \theta_1) = \Delta_A$, and so $\mathfrak{A}$ is not finitely-sibdirectly-irreducible. Otherwise, take any $\bar{b} \in A^2$ such that $c \triangleq (b_0 \wedge^{\mathfrak{A}} b_1) \notin (\operatorname{img}\bar{b})$, in which case, for each $j \in 2$, $\vartheta_j \triangleq ((\bigcup\{[c \wedge^{\mathfrak{A}} d, b_j \wedge^{\mathfrak{A}} d]^2_{\mathfrak{A}} \mid d \in A\}) \cup \Delta_A) \supsetneq \Delta_A$ is symmetric and forms a subalgebra of $\mathfrak{A}^2$, and so the transitive closure $\eta_j = \mathrm{Cg}^{\mathfrak{A}}(\{\langle c, b_j\rangle\}) \supseteq \vartheta_j$ of ϑ_j is a congruence of $\mathfrak{A}$ distinct from Δ_A. By contradiction, prove that $(\eta_0 \cap \eta_1) \subseteq \Delta_A$. For suppose $(\eta_0 \cap \eta_1) \nsubseteq \Delta_A$. Take any $\bar{e} \in ((\eta_0 \cap \eta_1) \setminus \Delta_A) \neq \varnothing$, in which case, for all $k, l \in 2$, $\langle e_k, e_{1-k}\rangle \in (\theta_l \setminus \Delta_A)$, that is, there are some $m_l \in \omega$, $\bar{f}^l \in A^{m_l+2}$ and $\bar{g}^l \in A^{m_l+1}$ such that $f^l_0 = e_k$, $f^l_{m_l+1} = e_{1-k}$ and, for every $n \in (m_l + 1)$, $f^l_{n[+1]} \in [c \wedge^{\mathfrak{A}} g^l_n, b_l \wedge^{\mathfrak{A}} g^l_n]_{\mathfrak{A}}$, and so $e_k \leqslant^{\mathfrak{A}} c$, when taking $n = 0$, because $\{l, 1-l\} = 2$, while $e_k = f^{l|(1-l)}_0 \leqslant^{\mathfrak{A}} (b_{l|(1-l)} \wedge^{\mathfrak{A}} g^{l|(1-l)}_0) \leqslant^{\mathfrak{A}} b_{l|(1-l)}$. By induction on any $\ell \in (m_l + 2)$, show that $e_k \leqslant^{\mathfrak{A}} f^l_\ell$. The case $\ell = 0$ is by the equality $e_k = f^l_0$. Otherwise, $(m_l + 2) \ni (\ell - 1) < \ell$, in which case, by induction hypothesis, we have $c \geqslant^{\mathfrak{A}} e_k \leqslant^{\mathfrak{A}} f^l_{\ell-1} \leqslant^{\mathfrak{A}} (b_l \wedge^{\mathfrak{A}} g^l_{\ell-1}) \leqslant^{\mathfrak{A}} g^l_{\ell-1}$, and so we get $e_k \leqslant^{\mathfrak{A}} (c \wedge^{\mathfrak{A}} g^l_{\ell-1}) \leqslant^{\mathfrak{A}} f^l_\ell$. In particular, $e_k \leqslant^{\mathfrak{A}} e_{1-k}$, when taking $\ell = (m_l + 1)$, since $f^l_{m_l+1} = e_{1-k}$. Then, $e_0 = e_1$, in which case this contradiction shows that $(\eta_0 \cap \eta_1) = \Delta_A$, and so $\mathfrak{A}$ is not finitely-sibdirectly-irreducible. Thus, by Lemma 2.3 as well as the simplicity of two-element algebras and absence of their proper non-one-element subalgebras, $(\mathrm{SI}^{(\omega)}\,|\,\mathrm{Si})(\mathsf{SL}) = \mathbf{I}\mathfrak{S}_2$ is the class of two-element semi-lattices, that is, the universal first-order model subclass of SL relatively axiomatized by the single universal first-order sentence $\forall_{\imath\in 3} x_\imath((x_2 \approx x_1) \vee (x_2 \approx x_0) \vee (x_1 \approx x_0))$, while SL, being finitely-semi-simple and finitely-generated, is semi-simple and locally-finite. On the other hand, since $\mathrm{Fi}(2) = \{\wp(N, 2) \mid N \subseteq 2\}$, the set $\{\Delta_{2^2}, (2^2)^2\} \cup \{\ker(\pi_\jmath\restriction 2^2) \mid \jmath \in 2\}$ of filtral congruences of $\mathfrak{S}^2_2$ does not contain its congruence $\Delta_{2^2} \cup \{\langle\langle 0, \Bbbk\rangle, \langle 0, 1 - \Bbbk\rangle\rangle \mid \Bbbk \in 2\}$, in which case, by Theorem 4.18, SL, not being directly filtral, is not {restricted} implicative, and so is not ⌈⟨sub⟩directly⌉ congruence-distributive ⌊in particular, is neither /finitely disjunctive, in view of Lemma 4.1, nor lattice expanding, in view of the well-known congruence-distributivity of lattice expansions; cf., e.g., [12]⌋. □

Likewise, the variety of Stone algebras demonstrates the necessity of the stipulation of relative [finite] semi-simplicity in the formulations of Corollaries 4.19, 4.20, 4.21, 4.22, 4.23 and Theorem 4.18 (cf. the next section).

4.3.1. *Restricted implicativity versus equality determinants.* Recall that a *(logical) Σ-matrix* is any pair $\mathcal{A} = \langle \mathfrak{A}, D^{\mathcal{A}} \rangle$ with its *underlying* Σ-algebra $\mathfrak{A}$ and its *truth predicate* $D^{\mathcal{A}} \subseteq A$ {in general, any Σ-matrix/"its underlying algebra" are denoted by same capital Calligraphic/Fraktur letter [with same indices, if any], its truth predicate being denoted by the letter D with superscript denoting the Σ-matrix}, in which case any $\Im \in \wp_\omega(\mathrm{Tm}^1_\Sigma)$ determines $\Im^{\mathcal{A}} : A \to \wp(\Omega), a \mapsto \{\iota \in \Im \mid \iota^{\mathfrak{A}}(a) \in D^{\mathcal{A}}\}$, and so is called a *[joint] equality/identity determinant for|of* a class M of Σ-matrices, if $\forall \mathcal{A} \in \mathsf{M}, \forall a, b \in A : (\forall \mathcal{B} \in ((\{\mathcal{A}\}[\cup \mathsf{M}]) \cap \pi_0^{-1}[\{\mathfrak{A}\}]) : \Im^{\mathcal{B}}(a) = \Im^{\mathcal{B}}(b)) \Rightarrow (a = b)$, being then a joint one ⟨cf. [16, 17, 18] for one-element M⟩.

Here, it is supposed that $\Sigma_+ \subseteq \Sigma$.

Given any $\bar{\varphi} \in (\mathrm{Tm}^1_\Sigma)^*$, $\iota \in \Im \in \wp_\omega(\mathrm{Tm}^1_\Sigma)$, $i \in 2$ and $\Delta \in \Xi \subseteq \wp(\mathrm{img}\,\bar{\varphi})$, let $\varepsilon^{i,\iota}_{\bar{\varphi},\Delta} \triangleq ((\wedge_+\langle(\bar{\varphi} \cap \Delta) * ((\bar{\varphi} \cap \Delta) \circ [x_0/x_1]), \iota(x_{2+i})\rangle) \lessapprox (\vee_+\langle(\bar{\varphi} \setminus \Delta) * ((\bar{\varphi} \setminus \Delta) \circ [x_0/x_1]), \iota(x_{3-i})\rangle)) \in \mathrm{Eq}^4_\Sigma$ and $\mho^{\bar{\varphi}}_{\Im,\Xi} \triangleq \{\varepsilon^{i,\iota}_{\bar{\varphi},\Delta} \mid i \in 2, \iota \in \Im \Delta \in \Xi\} \in \wp_\omega(\mathrm{Eq}^4_\Sigma)$.

Lemma 4.25. *Let M be a class of Σ-matrices, $\bar{\varphi} \in (\mathrm{Tm}^1_\Sigma)^*$, $\Omega \triangleq (\mathrm{img}\,\bar{\varphi})$, $\Xi \subseteq \wp(\Omega)$ and $\Im \in \wp_\omega(\mathrm{Tm}^1_\Sigma(\cap\Omega))$. Suppose, for all $\mathcal{A} \in \mathsf{M}$, $\{\Omega^{\mathcal{A}}[A] \subseteq \Xi$ and$\}$ $\mathfrak{A}{\upharpoonright}\Sigma_+$ is a [distributive] lattice $\{$while $D^{\mathcal{A}}$ is a prime filter of it, whereas $\Im$ is a joint equality determinant for $\mathsf{M}\}$. Then, $\mho^{\bar{\varphi}}_{\Im,\Xi}$ is an identity (reflexive) symmetric [transitive $\{$implication$\}$] system for $\pi_0[\mathsf{M}]$.*

Proof. Clearly, for all $j \in 2$, $\iota \in \Im$ and $\Delta \in \Xi$, there are some $\phi, \psi, \xi \in \mathrm{Tm}^3_\Sigma$ such that $(\varepsilon^{j,\iota}_{\bar{\varphi},\Delta}[x_3/x_2]) = ((\phi \wedge \xi) \lessapprox (\psi \vee \xi))$, in which case this is satisfied in lattice Σ-expansions, and so in $\pi_0[\mathsf{M}]$. (Likewise, there are then some $\bar{\eta}, \bar{\zeta} \in (\mathrm{Tm}^2_\Sigma)^+$ with $((\mathrm{img}\,\bar{\eta}) \cap (\mathrm{img}\,\bar{\zeta})) \neq \varnothing$ such that $(\varepsilon^{j,\iota}_{\bar{\varphi},\Delta}[x_{2+i}/x_i]_{i \in 2}) = ((\wedge_+\bar{\eta}) \lessapprox (\vee_+\bar{\eta}))$, in which case this is satisfied in lattice Σ-expansions, and so in $\pi_0[\mathsf{M}]$.) Furthermore, $(\mho^{\bar{\varphi}}_{\Im,\Xi}[x_2/x_3, x_3/x_2]) = \mho^{\bar{\varphi}}_{\Im,\Xi}$. [Next, since the Σ_+-quasi-identity $\{(x_0 \wedge x_1) \lessapprox (x_2 \vee x_3), (x_0 \wedge x_3) \lessapprox (x_2 \vee x_4)\} \to ((x_0 \wedge x_1) \lessapprox (x_2 \vee x_4))$, being satisfied in distributive lattices, is so in $\pi_0[\mathsf{M}]$, so are logical consequences of its substitutional Σ-instances $(\mho^{\bar{\varphi}}_{\Im,\Xi} \cup (\mho^{\bar{\varphi}}_{\Im,\Xi}[x_{2+i}/x_{3+i}]_{i \in 2})) \to \Psi$, where $\Psi \in (\mho^{\bar{\varphi}}_{\Im,\Xi}[x_3/x_4])$. {Finally, consider any $\mathcal{A} \in \mathsf{M}$, $a \in A$ and $\bar{b} \in (A^2 \setminus \Delta_A)$, in which case there are some $k \in 2$, $\iota \in \Im$ and $\mathcal{B} \in \mathsf{M}$ with $\mathfrak{B} = \mathfrak{A}$ such that $\iota^{\mathfrak{B}}(b_k) \in D^{\mathcal{B}} \not\ni \iota^{\mathfrak{B}}(b_{1-k})$, and so, as $\Delta \triangleq \Omega^{\mathcal{B}}(a) \in \Xi$, $\mathfrak{A} \not\models \varepsilon^{k,\iota}_{\bar{\varphi},\Delta}[x_i/a, x_{2+i}/b_i]_{i \in 2}$, for $D^{\mathcal{B}}$ is a prime filter of $\mathfrak{A}{\upharpoonright}\Sigma_+$.}] □

By the Prime Ideal Theorem, due to which V_1 is a joint equality determinant for any class M of Σ-matrices with underlying algebras, being lattice expansions, and truth predicates, being exactly *all* prime filters of the Σ_+-reducts of members of $\pi_0[\mathsf{M}]$, Lemma 4.25 immediately yields:

Corollary 4.26. *Let $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, $\bar{\varphi} \in (\mathrm{Tm}^1_\Sigma)^*$, $\Omega \triangleq (\mathrm{img}\,\bar{\varphi})$ and $\Im \in \wp_\omega(V_1, \mathrm{Tm}^1_\Sigma(\cap\Omega))$. Suppose, for all $\mathfrak{A} \in \mathsf{K}$, $\mathfrak{A}{\upharpoonright}\Sigma_+$ is a [distributive] lattice. Then, $\mho^{\bar{\varphi}}_{\Im,\wp(\Omega)}$ is an identity (reflexive) symmetric [transitive implication] system for $\pi_0[\mathsf{M}]$.*

This, in its turn, by Theorem 4.8, provides a practically immediate *constructive* insight/proof into/to REDPC for varieties of distributive|"De Morgan" lattices|"algebras||lattices", originally being due to [6]||[19]. And what is more, by Lemma 4.9, it immediately yields:

Corollary 4.27. *Let* $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, $\bar{\varphi} \in (\mathrm{Tm}^1_\Sigma)^*$, $\Omega \triangleq (\mathrm{img}\,\bar{\varphi})$ *and* $\Im \in \wp_\omega(V_1, \mathrm{Tm}^1_\Sigma)$. *Suppose, for all* $\mathfrak{A} \in \mathsf{K}$, $\mathfrak{A}{\restriction}\Sigma_+$ *is a distributive lattice, while* $\mho^{\bar{\varphi}}_{\Im,\wp(\Omega)}$ *is an implicative system for* K. *Then, members of* $\mathbf{IS}_{>1}\mathsf{K}$ *are simple.*

Theorem 4.28. *Let* M *be a class of* Σ*-matrices,* $\bar{\varphi} \in (\mathrm{Tm}^1_\Sigma)^*$, $\Omega \triangleq (\mathrm{img}\,\bar{\varphi})$ *and* $\Xi \subseteq | = \wp(\Omega)$. *Suppose, for all* $\mathcal{A} \in \mathsf{M}$, *"*$\Omega^{\mathcal{A}}[A] \subseteq \Xi$ *and"|* $\mathfrak{A}{\restriction}\Sigma_+$ *is a distributive lattice with set of its prime filters* $\{D^{\mathcal{B}} \mid \mathcal{B} \in \mathsf{M}, \mathfrak{B} = \mathfrak{A}\}$. *Then,* Ω *is an equality determinant for* M *iff* $\mho^{\bar{\varphi}}_{V_1,\Xi}$ *is an implicative system for* $(\mathbf{IS}_{[>1]})\pi_0[\mathsf{M}]$ *|([in which case its members are simple]).*

Proof. Let $\mathcal{A} \in \mathsf{M}$, $\bar{a} \in A^2$ and, for any $\bar{b} \in A^2$, $h_{\bar{b}} \triangleq [x_i/a_i, x_{2+i}/b_i]_{i \in 2}$. First, assume Ω is an equality determinant for M. Consider any $\bar{b} \in A^2$. Assume $\mathfrak{A} \not\models \varepsilon^{j,x_0}_{\bar{\varphi},\Delta}[h_{\bar{b}}]$, for some $j \in 2$ and $\Delta \in \Xi$, in which case, by the Prime Ideal Theorem, $\exists \mathcal{B} \in \mathsf{M} : \mathfrak{B} = \mathfrak{A}, \forall k \in 2 : \Delta = \Omega^{\mathcal{B}}(a_k)$, so $a_0 = a_1$. Moreover, by the Prime Ideal Theorem, V_1 is a joint equality determinant for M. Hence, by Lemma 4.25, $\mho^{\bar{\varphi}}_{V_1,\Xi}$ is an implicative system for $\mathfrak{A}$. Conversely, assume $\mho^{\bar{\varphi}}_{V_1,\Xi}$ is an implicative system for $\mathfrak{A}$ and $\Delta \triangleq \Omega^{\mathcal{A}}(a_0) = \Omega^{\mathcal{A}}(a_1)$. Take any $\bar{b} \in (D^{\mathcal{A}} \times (A \setminus D^{\mathcal{A}})) \neq \varnothing$, in which case, as $\Delta \in \Xi$, $\mathfrak{A} \not\models \varepsilon^{0,x_0}_{\bar{\varphi},\Delta}[h_{\bar{b}}]$, for $D^{\mathcal{A}}$ is a prime filter of $\mathfrak{A}{\restriction}\Sigma_+$, and so $a_0 = a_1$. |([Finally, Corollary 4.27 completes the argument.]) □

This, by Corollary 4.10, yields a new, quite transparent *constructive* proof/insight to/into REDPC for the [quasi-//pre-]variety of "distributive lattices"|"De Morgan algebras" (cf. [6]|[19]) generated by the |"diamond non-Boolean" one $\mathfrak{A}$ with carrier $A = 2^{|2}$ and zero‖unit $(0\|1)|\langle 0\|1, 0\|1\rangle$, when $\Sigma = (\Sigma_+ \cup (\varnothing|\{\neg, \bot, \top\}))$ |"with $(1 \wr 0)$-ary $\neg \wr (\bot, \top)$", $\mathsf{M} = (\{\mathfrak{A}\} \times (\{\{1\}\}|\{\{\langle 1,1\rangle, \langle i, 1-i\rangle\} \mid i \in 2\})$, $\bar{\varphi} = (\langle x_0\rangle * (\varnothing|\langle \neg x_0\rangle))$ and $\Xi = \wp(\Omega)$, as well as an immediate proof/insight to/into the well-known simplicity of non-one-element subalgebras of $\mathfrak{A}$, in its turn, by Corollaries 2.5 and 2.8, implying the equationality of the quasi-//pre-variety generated by $\mathfrak{A}$.

5. Strong Morgan-Stone lattices versus distributive lattices

Here, we deal with signatures $\Sigma^{(-)}_{+[,01]} \triangleq (\Sigma_+[\cup\{\top, \bot\}](\cup\{\neg\}))$. The variety of [bounded] distributive lattices is denoted by $[\mathsf{B}]\mathsf{DL} \subseteq \mathsf{A}_{\Sigma_{+[,01]}}$, that with carrier $n \in (\omega \setminus 1)$ and the natural ordering on this being denoted by $\mathfrak{D}_{n[,01]}$, in which case V_1 is an equality determinant for $\mathcal{D}_{2[,01]} \triangleq \langle \mathfrak{D}_{2[,01]}, \{1\}\rangle$, and so, as $\{1\}$ is a/"the only" prime filter of $\mathfrak{D}_{2[,01]}$, by [17, Lemma 11]/"Theorem 4.28 with $\mathsf{M} = \{\mathcal{D}_{2[,01]}\}$, $\bar{\varphi} = \langle x_0\rangle$ and $\Xi = \wp(V_1)$", $(\mho_+ \triangleq \{(x_i \wedge x_{2+j}) \lessapprox (x_{1-i} \wedge x_{3-j}) \mid i, j \in 2\})/\mho^{\langle x_0\rangle}_{V_1,\wp(V_1)}$ is a disjunctive/implicative system for $\mathfrak{D}_{2[,01]}$. Then, taking the Prime Ideal Theorem, Corollaries 4.10, 4.11 Lemmas 4.1, 4.9 and Remark 2.2 into account, we immediately have the following well-known fact (cf. [6] as to REDPC):

Lemma 5.1. *For any* $\mathfrak{A} \in [\mathsf{B}]\mathsf{DL}$ *and prime filter* F *of* $\mathfrak{A}[{\restriction}\Sigma_+]$, $h \triangleq \chi^F_A \in \hom(\mathfrak{A}, \mathfrak{D}_{2[,01]})$ *and* $h[A] = 2$, *in which case* $[\mathsf{B}]\mathsf{DL} = \mathbf{IP}^{\mathrm{SD}}\mathfrak{D}_{2[,01]}$, *and so* $[\mathsf{B}]\mathsf{DL}$ *is the finitely* $\mho_+$*-disjunctive restricted* $\mho^{\langle x_0\rangle}_{V_1,\wp(V_1)}$*-implicative congruence-distributive finitely-semi-simple (pre-/quasi-)variety generated by* $\mathfrak{D}_{2[,01]}$ *with* $(\mathrm{Si} \mid \mathrm{SI}^{\langle\omega\rangle})([\mathsf{B}]\mathsf{DL})$ $= \mathbf{I}\mathfrak{D}_{2[,01]} = ([\mathsf{B}]\mathsf{DL}_{\mho_+|\mho^{\langle x_0\rangle}_{V_1,\wp(V_1)}} \setminus \mathbf{I}(\mathfrak{D}^0_{2[,01]}))$ *and REDPC scheme* $\mho^{\langle x_0\rangle}_{V_1,\wp(V_1)}$.

A *[bounded/] strong (De-)Morgan-Stone lattice[/algebra]* is any $\Sigma^-_{+[,01]}$-algebra, whose $\Sigma_{+[,01]}$-reduct is a [bounded] distributive lattice and which satisfies the following Σ^-_+-identities:

$$\neg(x_0 \wedge x_1) \approx (\neg x_0 \vee \neg x_0), \tag{5.1}$$

$$x_0 \lessapprox \neg\neg x_0, \tag{5.2}$$

$$x_0 \gtrapprox (\neg\neg x_0 \wedge \neg x_0), \tag{5.3}$$

$$\neg\neg x_0 \lessapprox (x_0 \vee (\neg\neg x_1 \vee \neg x_1)), \tag{5.4}$$

in which case, by (5.1), (5.2) and (5.3), it satisfies the Σ^-_+-quasi-identities:

$$(x_0 \lessapprox x_1) \quad \rightarrow \quad (\neg x_1 \lessapprox \neg x_0), \tag{5.5}$$

$$(\neg x_0 \lessapprox x_0) \quad \leftarrow | \rightarrow \quad (\neg x_0 \lessapprox \neg\neg x_0), \tag{5.6}$$

and so the $\Sigma^-_{+[,01]}$-identities:

$$\neg(x_0 \vee x_1) \approx (\neg x_0 \wedge \neg x_1), \tag{5.7}$$

$$\neg\neg\neg x_0 \approx \neg x_0, \tag{5.8}$$

$$[\neg(\top|\bot) \approx (\bot|\top) \tag{5.9}$$

(in particular, it is a Morgan-Stone algebra; cf. [2, 21])], the variety of them being denoted by $[\mathsf{B}/]\mathsf{S}(\mathsf{D})\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$. Then, its members satisfying the Σ^-_+-identity:

$$\neg\neg x_0 \approx | \lessapprox x_0, \tag{5.10}$$

are exactly *[bounded/] De(-)Morgan lattices[/algebras]* (cf. [1, 14, 13]), the variety of them being denoted by $[\mathsf{B}/]\mathsf{DM}(\mathsf{L}[/\mathsf{A}])$. Likewise, *[bounded/] Stone lattices[/algebras]* are [bounded/] strong Morgan-Stone lattices[/algebras] satisfying the Σ^-_+-identity:

$$(x_0 \wedge \neg x_0) \lessapprox x_1, \tag{5.11}$$

[i.e., $(5.11)[(\lessapprox / \approx,)x_1/\bot]$], the variety of them being denoted by $[\mathsf{B}/]\mathsf{S}(\mathsf{L}[/\mathsf{A}])$. Then, members of $[\mathsf{B}/]\mathsf{B}(\mathsf{L}[/\mathsf{A}]) \triangleq ([\mathsf{B}]\mathsf{DML} \cap [\mathsf{B}]\mathsf{SL})$ are called *[bounded/] Boolean lattices[/algebras]*. Finally, *[bounded/] {strong} Kleene{-Stone} lattices[/algebras]* are [bounded] {strong} De-Morgan{-Stone} lattices satisfying the Σ^-_+-identity:

$$(x_0 \wedge \neg x_0) \lessapprox (x_1 \vee \neg x_1), \tag{5.12}$$

the variety of them being denoted by $[\mathsf{B}/]\{\mathsf{S}\}\mathsf{K}\{\mathsf{S}\}(\mathsf{L}[/\mathsf{A}])\{\supseteq [\mathsf{B}]\mathsf{SL}$, for $(5.12) = ((5.11)[x_1/(x_1 \vee \neg x_1)])\}$.

Let $\bar{\varphi} \triangleq \langle x_0, \neg x_0 \rangle$ and $\Omega \triangleq (\mathrm{img}\,\bar{\varphi})$.

Lemma 5.2. *Let $\mathfrak{A} \in [\mathsf{B}]\mathsf{SMSL}$, $a, b \in A$ and F a prime filter of $\mathfrak{A}{\upharpoonright}\Sigma_+$. Suppose both $((\neg^{\mathfrak{A}})a \in F) \Leftrightarrow ((\neg^{\mathfrak{A}})b \in F)$. Then, $(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a \in F) \Leftrightarrow (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b \in F)$.*

Proof. Assume $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a \in F$. If $b \in F$, then, as $\mathfrak{A} \models (5.2)[x_0/b]$, we have $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b \in F$. Otherwise, $a \notin F$, in which case, as $\mathfrak{A} \models (5.3)[x_0/a]$, we have $\neg^{\mathfrak{A}}a \notin F$, that is, $\neg^{\mathfrak{A}}b \notin F$, and so, since $\mathfrak{A} \models (5.4)[x_0/a, x_1/b]$, we get $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b \in F$ as well. □

This, by Theorem 4.8, the Prime Ideal one, (5.1), (5.7) and Corollary 4.26, immediately yields:

Corollary 5.3. *$\mho^{\bar{\varphi}}_{\Omega,\wp(\Omega)}$ is an REDPC scheme for $[\mathsf{B}]\mathsf{SMSL}$.*

This provides a uniform insight into REDPC for Stone and De Morgan algebras, originally given by separate distinct schemes in [9, 19].

Let $(\mathfrak{DM}|\mathfrak{S})_{(4|3)[,01]} \in [\mathsf{B}]\mathsf{SMSL}$ be the $\Sigma^-_{+[,01]}$-algebra with $((\mathfrak{DM}|\mathfrak{S})_{(4|3)[,01]}{\upharpoonright}\Sigma_{+[,01]}) \triangleq (\mathfrak{D}^2_{2[,01]}{\upharpoonright}(2^2 \setminus (\varnothing|\{\langle 1, 0\rangle\}))$ and $\neg^{(\mathfrak{DM}|\mathfrak{S})_{(4|3)[,01]}} \triangleq \{\langle\langle j, k\rangle, \langle 1 - (k|j), 1 - j\rangle\rangle \mid \langle j, k\rangle \in (DM|S)_{4|3}\}$ and $(((\mathfrak{K}\|\mathfrak{K}')/\mathfrak{B})_{(3/2)[,01]} \triangleq (\mathfrak{DM}_{4[,01]}{\upharpoonright}(\Delta_2 \cup (\{\langle 0\|1, 1\|0\rangle\}/\varnothing))$, in which case $\mu : K_3 \rightarrow K'_3, \langle \imath, \jmath\rangle \mapsto \langle \jmath, \imath\rangle$ is an isomorphism from $\mathfrak{K}_3$ onto $\mathfrak{K}'_3$, and so $\mathsf{S}_{[01]} \triangleq \{\mathfrak{DM}_{4[,01]}, \mathfrak{K}_{3[,01]}, \mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]}\}$, being a skeleton of $\mathsf{S}'_{[01]} \triangleq \{\mathfrak{DM}_{4[,01]}, \mathfrak{K}_{3[,01]}, \mathfrak{K}'_{3[,01]}, \mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]}\} = \mathbf{S}_{>1}\{\mathfrak{DM}_{4[,01]}, \mathfrak{S}_{3[,01]}\}$, is that of $\mathbf{I}\mathsf{S}'_{[01]} = \mathbf{I}\mathsf{S}_{[01]}$. Then, for any $\mathfrak{A} \in \mathsf{S}_{[01]}$, Ω is an equality determinant for $\langle \mathfrak{A}, A \cap \pi_0^{-1}[\{1\}]\rangle$, in which case, since $A \cap \pi_0^{-1}[\{1\}]$ is a prime filter of $\mathfrak{A}$, by [17, Lemma 11], $\mho_\Omega \triangleq \{(\tau(x_\Bbbk) \wedge \rho(x_{2+\ell})) \lessapprox (\tau(x_{1-\Bbbk}) \vee \rho(x_{3-\ell})) \mid \Bbbk, \ell \in 2, \tau, \rho \in \Omega\}$ is a finite disjunctive system for $\mathsf{S}_{[01]}$, and so, by Lemma 4.1, its members are subdirectly-irreducible, as it is well-known. And what is more, by the paragraph following the proof of Theorem 4.28, isomorphic copies of members of $\mathsf{S}''_{[01]} \triangleq \{\mathfrak{DM}_{4[,01]}, \mathfrak{K}_{3[,01]}, \mathfrak{K}'_{3[,01]}, \mathfrak{B}_{2[,01]}\} = \mathbf{S}_{>1}\mathfrak{DM}_{4[,01]}$, being $\mho^{\overline{\varphi}}_{V_1,\wp(\Omega)}$-implicative, are simple, as it is well-known. On the other hand,

$$\hbar \triangleq ((\pi_1 \restriction S_3) \times (\pi_1 \restriction S_3)) \in \hom(\mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]}), \tag{5.13}$$

in which case, by Lemma 2.3, since $\hbar(\langle 1,1\rangle) = \langle 1,1\rangle = \hbar(\langle 0,1\rangle) \neq \langle 0,0\rangle = \hbar(\langle 0,0\rangle)$, $\{\Delta_{S_3}, S_3^2\} \not\ni (\ker \hbar) = \hbar_2^{-1}[\Delta_{B_2}] \in \mathrm{Co}(\mathfrak{S}_{3[,01]})$, and so $\mathfrak{S}_{3[,01]}$ is not simple, as it is well-known. Thus, $\mathsf{S}''_{[01]} = \mathrm{Si}(\mathsf{S}'_{[01]})$. In particular, $\mathsf{S}'''_{[01]} \triangleq \{\mathfrak{DM}_{4[,01]}, \mathfrak{K}_{3[,01]}, \mathfrak{B}_{2[,01]}\} = \mathrm{Si}(\mathsf{S}_{[01]})$.

Theorem 5.4. *For any prime filter F of the Σ_+-reduct of any $\mathfrak{A} \in$ [B]SMSL, so is $G \triangleq (A \setminus (\neg^{\mathfrak{A}})^{-1}[F])$, in which case, for some $\mathfrak{B} \in \mathsf{S}'_{[01]}$, $h \triangleq (\chi^F_A \times \chi^G_A) \in \hom(\mathfrak{A}, \mathfrak{B})$ and $h[A] = B$, and so [B]SMSL $= \mathbf{IP}^{\mathrm{SD}}\mathsf{S}_{[01]}$. In particular, [B]SMSL is finitely $\mho_\Omega$-disjunctive with $([\mathsf{B}]\mathsf{SMSL}_{\mho_\Omega} \setminus \mathbf{I}(\prod \varnothing)) = \mathrm{SI}^{(\omega)}([\mathsf{B}]\mathsf{SMSL}) = \mathbf{IS}_{[01]} \ni \mathfrak{S}_{3[,01]} \notin \mathbf{IS}'''_{[01]} = \mathrm{Si}([\mathsf{B}]\mathsf{SMSL})$, in which case it is not \{finitely-\}semi-simple, and so is not $\langle$restricted$\rangle$ implicative.*

Proof. Take any $(a|b) \in (F|(A \setminus F)) \neq \varnothing$. Then, as $\mathfrak{A} \models (5.2|5.3)[x_0/(a|b)]$, we have $((\{\neg^{\mathfrak{A}}a\}|\{b, \neg^{\mathfrak{A}}b\}) \cap ((A \setminus G)|G)) \neq \varnothing$, in which case $G \neq \varnothing \neq (A \setminus G)$, and so, by (5.1) and (5.7), G is a prime filter of $\mathfrak{A}\restriction\Sigma_+$. Therefore, by (2.2), (2.4) and Lemma 5.1, h is a surjective homomorphism from $\mathfrak{A}\restriction\Sigma_{+[,01]}$ onto a subdirect square $\mathfrak{C}$ of $\mathfrak{D}_{2[,01]}$, in which case, for each $i \in 2$, as $\pi_i[C] = 2$, there are some $(a|b)_i \in C$ such that $\pi_i((a|b)_i) = (0|1)$, and so $C \supseteq \{a_0 \wedge^{\mathfrak{C}} a_1, b_0 \vee^{\mathfrak{C}} b_1\} = \Delta_2$. Then, there are some $(c|d) \in A$ such that $h(c|d) = \langle 0|1, 0|1\rangle$. And what is more, by Lemma 5.2, $(\ker h) \subseteq (\ker(\neg^{\mathfrak{A}} \circ h))$, in which case, by the Homomorphism Theorem, h is a surjective homomorphism from $\mathfrak{A}$ onto the $\Sigma^-_{+[,01]}$-algebra $\mathfrak{B}$ with $(\mathfrak{B}\restriction\Sigma_{+[,01]}) \triangleq \mathfrak{C}$ and $\neg^{\mathfrak{B}} \triangleq (h^{-1} \circ (\neg^{\mathfrak{A}} \circ h))$, and so $\mathfrak{B} \in$ [B]SMSL. Furthermore, $(c|d) \notin | \in F \ni | \not\ni \neg^{\mathfrak{A}}(c|d)$, in which case, as $\mathfrak{A} \models (5.3|5.2)[x_0/(c|d)]$, we have $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(c|d) \notin | \in F$, and so $\neg^{\mathfrak{B}}\langle 0|1, 0|1\rangle = h(\neg^{\mathfrak{A}}(c|d)) = \langle 1|0, 1|0\rangle$. In particular, $\mathfrak{B} = \mathfrak{B}_{2[0,1]}$, whenever $B = C = \Delta_2$. Next, if $\langle 1,0\rangle \in B = h[A]$, i.e., there is some $e \in A$ such that $h(e) = \langle 1,0\rangle$, viz., $e \in F \ni \neg^{\mathfrak{A}}e$, then, as $\mathfrak{A} \models (5.2)[x_0/e]$, we have $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}e \in F$, in which case $\neg^{\mathfrak{B}}\langle 1,0\rangle = h(\neg^{\mathfrak{A}}e) = \langle 1,0\rangle$, and so $\mathfrak{B} = \mathfrak{K}'_{3[0,1]}$, whenever $B = C = K'_3$. Likewise, if $\langle 0,1\rangle \in B = h[A]$, i.e., there is some $f \in A$ such that $h(f) = \langle 0,1\rangle$, viz., $f \notin F \not\ni \neg^{\mathfrak{A}}f$, then $\neg^{\mathfrak{B}}\langle 0,1\rangle = h(\neg^{\mathfrak{A}}f) = \langle 0,m\rangle$, for some $m \in 2$, and so $\mathfrak{B} = (\mathfrak{S}\|\mathfrak{K})_{3[0,1]}$, whenever $B = C = K_3 = S_3$ and $m = (0\|1)$. Finally, if $B = C = 2^2$, then, since $\mathfrak{B} \models (5.1)[x_0/\langle 1,0\rangle, x_1/\langle 0,1\rangle]$, we have $\langle 1,1\rangle = \neg^{\mathfrak{B}}\langle 0,0\rangle = \neg^{\mathfrak{B}}(\langle 1,0\rangle \wedge^{\mathfrak{B}} \langle 0,1\rangle) = (\neg^{\mathfrak{B}}\langle 1,0\rangle \vee^{\mathfrak{B}} \neg^{\mathfrak{B}}\langle 0,1\rangle) = (\langle 1,0\rangle \vee^{\mathfrak{B}} \langle 0,m\rangle) = \langle 1,m\rangle$, in which case $m = 1$, and so $\mathfrak{B} = \mathfrak{DM}_{4[0,1]}$. Thus, in any case, $\mathfrak{B} \in \mathsf{S}'_{[01]}$. In this way, the Prime Ideal Theorem, Corollaries 2.4, 2.8, Lemmas 2.3, 4.1, 4.9 and Remark 2.2 complete the argument. $\square$

Corollary 5.5. *Sub-varieties of [B]SMSL form the non-chain distributive seven-element lattice, whose Hasse diagram is depicted at Figure 1, where any (non-)solid circle-node is marked by $\mathsf{V} : S$ with a variety $\mathsf{V} \subseteq$ [B]SMSL, (not) being $(\omega|\infty)$-semi-simple/"\{$\langle$sub$\rangle$direecttly\} filtral"/"$\lceil$restricted $\lfloor\mho^{\overline{\varphi}}_{V_1\|\Omega,\wp(\Omega)}$-$\rfloor\rceil$implicative", and the least $S \subseteq \mathsf{S}_{[01]}$ such that $(\mathsf{V} \cap \mathsf{S}_{[01]}) = \mathbf{S}_{>1}S$, in which case $\mathrm{SI}(\mathsf{V}) = \mathbf{IS}_{>1}S$, while $\mathsf{V} = \mathbf{ISP}S$ is finitely $\mho_\Omega$-disjunctive, and so disjunctive//"restricted implicative"*

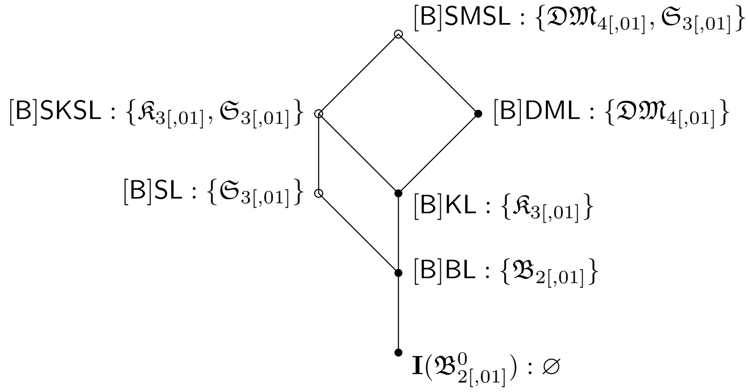

FIGURE 1. The lattice of sub-varieties of [B]SMSL.

sub-pre-//quasi-varieties of [B]SMSL *are sub-varieties of* [B]SMSL//[B]DML*. In particular:*

(i) [B]SMSL *is the (pre-/quasi-)variety generated by* $\{\mathrm{SI}\}$([B]DML) $\cup$ $\{\mathrm{SI}\}$([B]SL)*;*
(ii) [B]SKSL *is the sub-variety of* [B]SMSL *relatively axiomatized by* (5.12)$[x_1/\neg x_1]$*.*

Proof. Clearly, [B]DML $\ni \mathfrak{DM}_{4[,01]} \not\models ((5.12)[x_1/(\neg)x_1])[x_i/\langle i, 1-i\rangle]_{i\in 2}$, in which case $(\mathsf{S}_{[01]} \cap \mathrm{Mod}((5.12)[x_1/(\neg)x_1])) = (\mathsf{S}_{[01]} \setminus \{\mathfrak{DM}_{4[,01]}\})$, and so (ii) is due to Theorem 5.4, while [B]SL $\ni \mathfrak{S}_{3[,01]} \not\models (5.10)[x_0/\langle 0,1\rangle]$, whereas [B]KL $\ni \mathfrak{K}_{3[,01]} \not\models (5.11)[x_0/\langle 0, 1-i\rangle]_{i\in 2}$, as well as $\mathfrak{B}_{2[,01]} \in$ [B]BL is not one-element, in which case the seven varieties involved do form the lattice with Hasse diagram depicted at Figure 1 and their intersections with $\mathsf{S}_{[01]}$ exhausting all those subsets of this, which are closed under $\mathbf{S}_{>1}$, and so Theorems 4.18, 4.28 with $\mathsf{M} = (\{\mathfrak{DM}_{4[,01]}\} \times \{2^2 \cap \pi_j^{-1}[\{1\}] \mid j \in 2\})$, 5.4, Corollaries 4.10, 4.11, 5.3 as well as Lemmas 4.1 and 4.9 complete the argument. □

It is in the sense of (i) that [B]SMSL is the implicational/(quasi-)equational join of [B]DML and [B]SL.

Let $\mathfrak{K}_{(i:)4[,01]}$ (where $i \in \{1,2\}$) be the $\Sigma^-_{+[,01]}$-algebra with $(\mathfrak{K}_{(i:)4[,01]} \restriction \Sigma_{+[,01]}) \triangleq \mathfrak{D}_{4[,01]}$ and $\neg^{\mathfrak{K}_{(i:)4[,01]}} \triangleq (\{\langle j, 3-j\rangle \mid j \in (4(\setminus\{1,2\}))\}(\cup\{\langle i,i\rangle, \langle 3-i, 2\cdot(i-1)\rangle\}))$ (in which case, providing $i = (1|2)$, it satisfies (5.1), (5.2) and (5.3)|(5.4) [as well as (5.9) {in particular, it is a Morgan-Stone algebra; cf. [2, 21]}], but (5.4)|(5.3) is not true in it under $[x_k/(2-k)]_{k\in 2}|[x_0/1]$, and so neither (5.3) nor (5.4) can be omitted).

5.1. **Pre-varieties of strong Morgan-Stone lattices.**

Definition 5.6 (cf. [14])**.** Members of [B]{S}(DM‖K){S})L, satisfying the following Σ^-_+-quasi-identity:

$$(\neg x_0 \approx x_0) \to (x_0 \approx x_1), \tag{5.14}$$

are called *non-idempotent*, NI[B/]{S}((D)M‖K){S}(L[/A]) denoting their quasi-variety {and including [B]SL}. □

Lemma 5.7. *Any (non-one-element finitely-generated)* $\mathfrak{A} \in$ [B]SMSL *is non-idempotent if(f)* $\hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) \neq \varnothing$*, in which case* ([B]SMSL $\setminus$ [B]NISMSL) $\subseteq$ [B]DML*, and so* [B]SMSL $=$ ([B]NISMSL $\cup$ [B]DML)*.*

Proof. The "if" part is by the fact that $\neg^{\mathfrak{B}_{2[,01]}} a = a$, for no $a \in \Delta_2$. (Conversely, assume $\mathfrak{A}$ is non-idempotent, in which case, if $\hom(\mathfrak{A}, \mathfrak{S}_{3[,01]})$ is non-empty, then so is $\hom(\mathfrak{A}, \mathfrak{B}_{2[,01]})$, in view of (5.13). Otherwise, by Remark 2.2 and Corollary 5.5, $\mathfrak{A}[\upharpoonright\Sigma_+^-]$ [being finitely-generated, as $\{\bot^{\mathfrak{A}}, \top^{\mathfrak{A}}\}$ is finite] is a non-one-element finitely-generated non-idempotent De Morgan lattice. Then, by [14, Lemma 4.3], [since $\mathfrak{B}_2$ has no proper subalgebra, in which case, for each $h \in \hom(\mathfrak{A}\upharpoonright\Sigma_+^-, \mathfrak{B}_2)$, there are some $(a|b) \in A$ such that $h(a|b) = \langle 0|1, 0|1\rangle$, and so $h((\bot|\top)^{\mathfrak{A}}) = h((a|b)(\wedge|\vee)^{\mathfrak{A}}(\bot|\top)^{\mathfrak{A}}) = \langle 0|1, 0|1\rangle$] $\varnothing \neq \hom(\mathfrak{A}[\upharpoonright\Sigma_+^-], \mathfrak{B}_2)[= \hom(\mathfrak{A}, \mathfrak{B}_{2,01})]$.) Finally, Remark 2.2, Corollary 5.5 and (5.13) complete the argument. $\Box$

This, by Remark 2.2, Corollary 5.5, (2.1), (2.4) and the locality of quasi-varieties, immediately yields:

Corollary 5.8. $\mathsf{NI[B]\{S\}(DM|K)\{S\}L}$ *is the pre-/quasi-variety generated by* $\{(\mathfrak{DM}|\mathfrak{K})_{(4|3)[,01]} \times \mathfrak{B}_{2[,01]}\{, \mathfrak{S}_{3[,01]}\}\}$. *In particular, any (non-one-element)* $\mathfrak{A} \in \mathsf{[B]SMSL}$ *is non-idempotent if(f)* $\hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) \neq \varnothing$.

Likewise, Lemma 5.7 and [14, Proof of Lemma 4.9] immediately yield:

Corollary 5.9. $\mathfrak{K}_3$ *is embeddable into any member of* $\mathsf{SKSL} \setminus \mathsf{NISKSL}$.

Corollary 5.10. $\mathsf{NI[B]\{S\}DM\{S\}L} \cup \mathsf{[B]\{S\}K\{S\}L}$ *is the sub-quasi-variety of* $\mathsf{[B]\{S\}DM\{S\}L}$ *relatively axiomatized by the* Σ_+^-*-quasi-identity:*

$$(\neg x_0 \approx x_0) \to (x_0 \lessapprox (x_1 \vee \neg x_1)) \tag{5.15}$$

and is the pre-/quasi-variety generated by $\{\mathfrak{DM}_{4[,01]} \times \mathfrak{B}_{2[,01]}, \mathfrak{K}_{3[,01]}\{, \mathfrak{S}_{3[,01]}\}\}$.

Proof. Clearly, (5.15) is satisfied in $\mathsf{NI[B]\{S\}DM\{S\}L} \cup \mathsf{[B]\{S\}K\{S\}L}$. Conversely, consider any $\mathfrak{A} \in (\mathsf{[B]\{S\}DM\{S\}L} \setminus \mathsf{NI[B]\{S\}DM\{S\}L})$ satisfying (5.15) and any $a, b \in A$, in which case there is some $c \in A$ such that $\neg^{\mathfrak{A}} c = c$, and so, as $\mathfrak{A}(5.15)[x_0/c, x_1/(a|b)]$, we have $c \leqslant^{\mathfrak{A}} ((a|b) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}(a|b))$. Then, by (5.2), (5.5) and (5.7), we get $(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \leqslant^{\mathfrak{A}} c$, in which case $\mathfrak{A} \models (5.12)[x_0/a, x_1/b]$, and so $\mathfrak{A} \in \mathsf{[B]\{S\}K\{S\}L}$. Thus, Corollaries 5.5 and 5.8 complete the argument. $\Box$

This, by Lemma 5.7 and [14, Case 8 of Proof of Theorem 4.8], immediately yields:

Corollary 5.11. $\mathfrak{DM}_4$ *is embeddable into any member of* $\mathsf{\{S\}DM\{S\}L}$ *not satisfying* (5.15).

Members of $\mathsf{[B]\{S\}K\{S\}L}$ satisfying the Σ_+^--quasi-identity:

$$\{\neg x_0 \lessapprox x_0, (x_0 \wedge \neg x_1) \lessapprox (\neg x_0 \vee x_1)\} \to (\neg x_1 \lessapprox x_1) \tag{5.16}$$

are called *regular/classical* (cf. [14]/[15]), the quasi-variety of them being denoted by $\mathsf{R[B/]\{S\}K\{S\}(L[/A])}$.

Lemma 5.12. $\{\mathfrak{K}_{4[,01]}\{, \mathfrak{S}_{3[,01]}\}\} \subseteq \mathsf{R[B]\{S\}K\{S\}L} \subseteq \mathsf{NI[B]\{S\}K\{S\}L}$.

Proof. The fact that $\mathfrak{K}_{4[,01]} \in \mathsf{[B]KL}\{\subseteq \mathsf{[B]SKSL}\}$ is well-known, while its regularity is by the fact that $F \triangleq \{\neg^{\mathfrak{K}_{4[,01]}} i \leqslant^{\mathfrak{K}_{4[,01]}} i \mid i \in 4\} = (4 \setminus 2) = \neg^{\mathfrak{K}_{4[,01]}}[4 \setminus F]$ is a prime filter of $\mathfrak{D}_4$ {whereas that of $\mathfrak{S}_{3[,01]} \in \mathsf{[B]SKSL}$ is immediate}. For proving the second inclusion, consider any $\mathfrak{A} \in \mathsf{R[B]\{S\}K\{S\}L}$ and any $a, b \in A$ such that $\neg^{\mathfrak{A}} a = a$, in which case, as $\mathfrak{A} \models (5.1|5.16)[x_0/a, x_1/((\neg^{\mathfrak{A}})b|(a \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}})b))]$ (and $\mathfrak{A} \models (5.2)[x_0/b]$), we have $(b \leqslant^{\mathfrak{A}})\neg^{\mathfrak{A}}(\neg^{\mathfrak{A}})b \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}(\neg^{\mathfrak{A}})b) = \neg^{\mathfrak{A}}(a \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}})b) \leqslant^{\mathfrak{A}} (a \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}})b) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}})b$, and so $\neg^{\mathfrak{A}} b = b$. Then, since $\mathfrak{A} \models (5.12)[x_0/(a|b), x_1/(b|a)]$, we eventually get both $a(\leqslant | \geqslant)^{\mathfrak{A}} b$, i.e., $a = b$. Thus, $\mathfrak{A}$ is non-idempotent. $\Box$

Corollary 5.13. $\mathfrak{K}_4$ *is embeddable into any* $\mathfrak{A} \in (\mathsf{NISKSL} \setminus \mathsf{SL}) \supseteq (\mathsf{RSKSL} \setminus \mathsf{SL})$.

Proof. Then, there are some $a, b \in A$ such that $c \triangleq (a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \neq d \triangleq (b \wedge^{\mathfrak{A}} c) \leqslant^{\mathfrak{A}} c$, in which case, applying (5.1) and (5.5) [twice], we have $[\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} d \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}]c \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} c \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} d$, and so, by (5.2) and (5.3), we get $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(c|d) = (c|d)$. In this way, as $c \neq d$, by (5.14), we have $\neg^{\mathfrak{A}} c \neq c$, in which case we get $\neg^{\mathfrak{A}} d \neq \neg^{\mathfrak{A}} c$, and so $\{\langle 0, d\rangle, \langle 1, c\rangle, \langle 2, \neg^{\mathfrak{A}} c\rangle, \langle 3, \neg^{\mathfrak{A}} d\rangle\}$ is an embedding of $\mathfrak{K}_4$ into $\mathfrak{A}$. Finally, Lemma 5.12 completes the argument. ☐

Theorem 5.14. $\mathsf{R[B]\{S\}K\{S\}L}$ *is the pre-/quasi-variety generated by* $\{\mathfrak{K}_{4[,01]}\{, \mathfrak{S}_{3[,01]}\}\}$.

Proof. In view of locality of quasi-varieties and Lemma 5.12, it suffices to prove that any finitely-generated non-one-element $\mathfrak{A} \in \mathsf{R[B]\{S\}K\{S\}L}$ belongs to the pre-variety generated by $\{\mathfrak{K}_{4[,01]}\{, \mathfrak{S}_{3[,01]}\}\}$. Assume $\mathfrak{A}$ is generated by $\operatorname{img} \bar{a}$, for some $\bar{a} \in A^n$ and some $n \in (\omega \setminus 1)$. Put $b \triangleq (\wedge_{+}^{\mathfrak{A}} \langle a_m \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a_m\rangle_{m \in n})$, in which case, by (5.1), (5.7) and (5.12), we have $\neg^{\mathfrak{A}} b \leqslant^{\mathfrak{A}} b$. Consider any $h \in \hom(\mathfrak{A}, \mathfrak{K}_{3[,01]})$. Let $(I|J) \triangleq \{i \in n \mid h(a_i) = \langle 0|1, 0|1\rangle\}$, $(\imath|\jmath) = |(I|J)|$ and $\bar{\Bbbk}|\bar{\ell}$ any bijection from $\imath|\jmath$ onto $I|J$. We prove, by contradiction, that there is some $g \in \hom(\mathfrak{A}, \mathfrak{B}_{2[,01]})$ such that $g[\operatorname{img}((\bar{\Bbbk}|\bar{\ell}) \circ \bar{a})] = \{\langle 0|1, 0|1\rangle\}$. For suppose that, for every $g \in \hom(\mathfrak{A}, \mathfrak{B}_{2[,01]})$, there is either some $i \in \imath$ or some $j \in \jmath$ such that $g(a_{(\Bbbk|\ell)_{i|j}}) = \langle 1|0, 1|0\rangle$, in which case, as, by Lemmas 5.7 and 5.12, we have $\hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) \neq \varnothing$, we get $(I \cup J) \neq \varnothing$, and so we are allowed to put $c \triangleq (\vee_{+}^{\mathfrak{A}}((\bar{\Bbbk} \circ \bar{a} \circ \neg^{\mathfrak{A}} \circ \neg^{\mathfrak{A}}) * (\bar{\ell} \circ \bar{a} \circ \neg^{\mathfrak{A}})))$. Then, $\neg^{\mathfrak{A}} c \nleqslant^{\mathfrak{A}} c$, for $h(c) = \langle 0, 0\rangle \ngeqslant^{\mathfrak{K}_{3[,01]}} \langle 1, 1\rangle = h(\neg^{\mathfrak{A}} c)$. Now, consider any $f \in (\hom(\mathfrak{A}, \{\mathfrak{K}_{3[,01]}, \mathfrak{S}_{3[,01]}\})$ and the following complementary cases:

- $f \in \hom(\mathfrak{A}, \mathfrak{S}_{3[,01]})$,
  in which case, by (5.13) and the assumption to be disproved, $f(c) = \langle 1, 1\rangle$, and so $f(b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} c) = \langle 0, 0\rangle \leqslant^{\mathfrak{S}_{3[,01]}} \langle 1, 1\rangle = f(\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} c)$.
- $f \notin \hom(\mathfrak{A}, \mathfrak{S}_{3[,01]})$,
  in which case $f \in \hom(\mathfrak{A}, \mathfrak{K}_{3[,01]})$, while, as $\mathfrak{B}_{2[,01]}$ is a subalgebra of both $\mathfrak{K}_{3[,01]}$ and $\mathfrak{S}_{3[,01]}$, there is some $i \in n$ such that $f(a_i) = \langle 0, 1\rangle$, and so $f(b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} c) \leqslant^{\mathfrak{K}_{3[,01]}} f(b) = \langle 0, 1\rangle = f(\neg^{\mathfrak{A}} b) \leqslant^{\mathfrak{K}_{3[,01]}} f(\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} c)$.

In this way, since, by Corollary 5.5, $\mathsf{[B]SKSL} \ni \mathfrak{A}$ is the pre-variety generated by $\{\mathfrak{K}_{3[,01]}, \mathfrak{S}_{3[,01]}\}$, by Remark 2.2, we eventually get $(b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} c) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} c)$, in which case $\mathfrak{A} \not\models (5.16)[x_0/b, x_1/c]$, and so this contradiction to the regularity of $\mathfrak{A}$ definitely shows existence of some $g \in \hom(\mathfrak{A}, \mathfrak{B}_{2[,01]})$ such that $g[\operatorname{img}((\bar{\Bbbk}|\bar{\ell}) \circ \bar{a})] = \{\langle 0|1, 0|1\rangle\}$. Then, by (2.4), $h' \triangleq (h \times g) \in \hom(\mathfrak{A}, \mathfrak{K}_{3[,01]} \times \mathfrak{B}_{2[,01]})$, while $e \triangleq \{\langle k, \langle [\frac{k}{3}], \min(k, 1), \langle [\frac{k}{2}], [\frac{k}{2}]\rangle\rangle\rangle \mid k \in 4\} \in \hom(\mathfrak{K}_{4[,01]}, \mathfrak{K}_{3[,01]} \times \mathfrak{B}_{2[,01]})$ is injective, whereas $(\operatorname{img} h') \subseteq (\operatorname{img} e)$, in which case $h'' \triangleq (h' \circ e^{-1}) \in \hom(\mathfrak{A}, \mathfrak{K}_{4[,01]})$ as well as, by (2.1) and the injectivity of e^{-1}, $(\ker h'') = (\ker h') \subseteq (\ker h)$, and so Corollary 5.5 and Remark 2.2 complete the argument. ☐

Lemma 5.15. $\mathfrak{K}_3 \times \mathfrak{B}_2$ *is embeddable into any* $\mathfrak{A} \in (\mathsf{NISKSL} \setminus \mathsf{RSKSL})$.

Proof. Then, by (5.1), (5.5), (5.6), (5.7) and (5.8), there are some $a, b \in A$ such that $(c|d) \triangleq \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(a|b)(\geqslant \mid \ngeqslant)^{\mathfrak{A}}\neg^{\mathfrak{A}}(c|d)$ and $(c \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} d) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} c \vee^{\mathfrak{A}} d)$, in which case, using (5.1), (5.7) and (5.8), by induction on construction of any $\varphi \in \mathrm{Tm}^2_{\Sigma^-_+}$, we get $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\varphi^{\mathfrak{A}}(c, d) = \varphi^{\mathfrak{A}}(c, d)$, and so the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{c, d\}$ is a non-idempotent Kleene lattice such that $\mathfrak{B} \not\models (5.16)[x_0/c, x_1/d]$. Hence, by [14, Case 4 of Proof of Theorem 4.8], $\mathfrak{K}_3 \times \mathfrak{B}_2$ is embeddable into $\mathfrak{B}$, and so into $\mathfrak{A}$. ☐

Lemma 5.16. $\mathfrak{DM}_4 \times \mathfrak{B}_2$ *is embeddable into any* $\mathfrak{A} \in (\mathsf{NISMSL} \setminus \mathsf{SKSL})$.

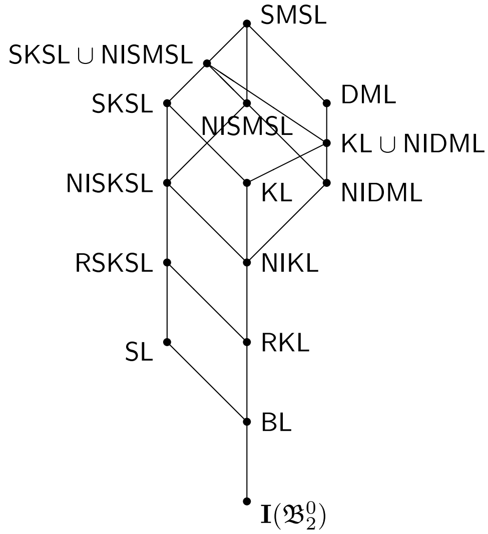

FIGURE 2. The lattice of sub-pre-/quasi-varieties of SMSL.

Proof. Then, taking Corollary 5.5(ii) into account, there are some $a, b \in A$ such that, by (5.2), $c \triangleq \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \nleqslant^{\mathfrak{A}} d \triangleq (\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} b)$, in which case, by (5.1), (5.7) and (5.8), we have both $\neg^{\mathfrak{A}}(c|d)(\geqslant | \leqslant)^{\mathfrak{A}}(c|d) = \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(c|d)$, and so, by induction on construction of any $\varphi \in \mathrm{Tm}^2_{\Sigma^-_+}$, we get $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\varphi^{\mathfrak{A}}(c, d) = \varphi^{\mathfrak{A}}(c, d)$. Thus, the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{c, d\}$ is a non-idempotent De Morgan lattice such that $\mathfrak{B} \not\models (5.12)[x_0/c, x_1/d]$, in which case, by the proof of [14, Lemma 4.10], $\mathfrak{DM}_4 \times \mathfrak{B}_2$ is embeddable into $\mathfrak{B}$, and so into $\mathfrak{A}$. □

Lemma 5.17. *$\mathfrak{S}_3$ is embeddable into any $\mathfrak{A} \in (\mathsf{SMSL} \setminus \mathsf{DML})$.*

Proof. Then, there is some $a \in A$ such that $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \neq a$, in which case, by (5.2), $b \triangleq (\neg^{\mathfrak{A}} a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a) \leqslant^{\mathfrak{A}} c \triangleq (a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \leqslant^{\mathfrak{A}} d \triangleq (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a)$, while, by (5.1), (5.7) and (5.8), both $\neg^{\mathfrak{A}} c = b = \neg^{\mathfrak{A}} d$ and $\neg^{\mathfrak{A}} b = d$, whereas $c \neq d$, for, otherwise, since $\mathfrak{A} \models (5.2|5.3)[x_0/a]$, $\{b, \neg^{\mathfrak{A}} a, a, \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a, d\}$ would be a pentagon of the distributive lattice $\mathfrak{A}\restriction\Sigma_+$, and so $b \neq c$, for otherwise, we would have $c = b = \neg^{\mathfrak{A}} c = \neg^{\mathfrak{A}} b = d$. Thus, $\{\langle 0, 0, b\rangle, \langle 0, 1, c\rangle, \langle 1, 1, d\rangle\}$ is an embedding of $\mathfrak{S}_3$ into $\mathfrak{A}$. □

Theorem 5.18. *Sub-pre/quasi-varieties of* SMSL *form the fifteen-element non-chain distributive lattice depicted at Figure 2.*

Proof. We use Corollary 5.5 tacitly. Clearly, $\mathfrak{DM}_4 \times \mathfrak{B}_2$ is not in SKSL, for $\mathfrak{DM}_4$ is not so, while $\pi_0\restriction(2^2 \times \Delta_2)$ is a surjective homomorphism from the former onto the latter, in which case, by Corollary 5.10, SKSL $\subsetneq$ (SKSL ∪ NISMSL) $\subsetneq$ SMSL, for SMSL $\ni \mathfrak{DM}_4 \not\models (5.15)[x_i/\langle i, 1-i\rangle]_{i \in 2}$. Likewise, $\mathfrak{S}_3 \notin$ DML, so, by Corollaries 5.8, 5.10 and Theorem 5.14, both (KL ∪ NIDML) $\subsetneq$ (SKSL ∪ NISMSL), NIDML $\subsetneq$ NISMSL, NIKL $\subsetneq$ NISKSL and RKL $\subsetneq$ RSKSL, while, by Corollary 5.8, NIKL $\ni (\mathfrak{K}_3 \times \mathfrak{B}_2) \not\models (5.16)[x_0/\langle\langle 0, 1\rangle, \langle 1, 1\rangle\rangle, x_1/(\langle\langle 0, 0\rangle, \langle 1, 1\rangle\rangle]$, so, by Lemma 5.12, RSKSL $\subsetneq$ NISKSL, whereas KL $\ni \mathfrak{K}_3 \not\models (5.14)[x_0/\langle 0, 1\rangle, x_1/\langle 0, 0\rangle]$, so NISKSL $\subsetneq$ SKSL. Finally, by Theorem 5.14, $\mathfrak{S}_3 \in$ RSKSL $\ni \mathfrak{K}_4 \not\models (5.11)[x_i/(1-i)]_{i \in 2}$, so SL $\subsetneq$ RSKSL. Thus, by Lemma 5.7, Corollaries 5.8, 5.10, Theorem 5.14 and [14, Theorem 4.8], the fifteen quasi-varieties involved are pair-wise distinct and do form the lattice depicted at Figure 2. Now, consider any pre-variety P $\subseteq$ SMSL such that P $\nsubseteq$ DML, in which case, by Lemma 5.17, $\mathfrak{S}_3 \in$ P, and so SL $\subseteq$ P, as well as the following exhaustive cases:

(1) $\mathsf{P} \nsubseteq (\mathsf{SKSL} \cup \mathsf{NISMSL})$,
in which case, by Corollaries 5.10 and 5.11, $\mathfrak{DM}_4 \in \mathsf{P} \ni \mathfrak{S}_3$, and so $\mathsf{P} = \mathsf{SMSL}$.
(2) $\mathsf{P} \subseteq (\mathsf{SKSL} \cup \mathsf{NISMSL})$ but neither $\mathsf{P} \subseteq \mathsf{SKSL}$ nor $\mathsf{P} \subseteq \mathsf{NISMSL}$,
in which case $(\mathsf{SKSL}|\mathsf{NISMSL}) \not\supseteq (\mathsf{P} \cap (\mathsf{NISMSL}|\mathsf{SKSL}))$, and so, by Lemma|Corollary 5.16|5.9 $((\mathfrak{DM}_4 \times \mathfrak{B}_2)|\mathfrak{K}_3) \in \mathsf{P} \ni \mathfrak{S}_3$. Then, by Corollary 5.10, $\mathsf{P} = (\mathsf{SKSL} \cup \mathsf{NISMSL})$.
(3) $\mathsf{P} \subseteq \mathsf{NISMSL}$ but $\mathsf{P} \nsubseteq \mathsf{SKSL}$,
in which case, by Lemma 5.16, $(\mathfrak{DM}_4 \times \mathfrak{B}_2) \in \mathsf{P} \ni \mathfrak{S}_3$, and so, by Corollary 5.8, $\mathsf{P} = \mathsf{NISMSL}$.
(4) $\mathsf{P} \subseteq \mathsf{SKSL}$ but $\mathsf{P} \nsubseteq \mathsf{NISMSL}$,
in which case, by Corollary 5.9, $\mathfrak{K}_3 \in \mathsf{P} \ni \mathfrak{S}_3$, and so $\mathsf{P} = \mathsf{SKSL}$.
(5) $\mathsf{P} \subseteq \mathsf{NISKSL}$ but $\mathsf{P} \nsubseteq \mathsf{RSKSL}$,
in which case, by Lemma 5.15, $(\mathfrak{K}_3 \times \mathfrak{B}_2) \in \mathsf{P} \ni \mathfrak{S}_3$, and so, by Corollary 5.8, $\mathsf{P} = \mathsf{NISKSL}$.
(6) $\mathsf{P} \subseteq \mathsf{RSKSL}$ but $\mathsf{P} \nsubseteq \mathsf{SL}$,
in which case, by Corollary 5.13, $\mathfrak{K}_4 \in \mathsf{P} \ni \mathfrak{S}_3$, and so, by Theorem 5.14, $\mathsf{P} = \mathsf{RSKSL}$.
(7) $\mathsf{P} \subseteq \mathsf{SL}$,
in which case $\mathsf{P} = \mathsf{SL}$.

In this way, [14, Theorem 4.8] completes the argument. □

5.1.1. *Implicative quasi-varieties of strong Morgan-Stone lattices and algebras.*

Lemma 5.19. *Let* $\mathsf{Q} \subseteq [\mathsf{B}]\mathsf{SMSL}$ *be a quasi-variety. Then,* $(\mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}) \cap [\mathsf{B}]\mathsf{NISMSL}) \subseteq \mathbf{I}\mathfrak{B}_{2[0,1]} \subseteq [\mathsf{B}]\mathsf{BL} \subseteq [\mathsf{B}]\mathsf{KL} \subseteq [\mathsf{B}]\mathsf{DML}$.

Proof. Consider any $\mathfrak{A} \in (\mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}) \cap [\mathsf{B}]\mathsf{NISMSL})$, in which case $|1A| > 1$ [viz., $\bot^{\mathfrak{A}} \neq \top^{\mathfrak{A}}$], i.e., $\mathfrak{A} \notin \mathbf{I}\mathfrak{B}^0_{2[,01]}$, and so [as, by (5.9), $\{\langle 0,0,\bot^{\mathfrak{A}}\rangle, \langle 1,1,\top^{\mathfrak{A}}\rangle\} \in \hom(\mathfrak{B}_{2,01}, \mathfrak{A})$ is injective], by Corollary 5.5 and Theorem 5.18, $\mathfrak{B}_{2[,01]} \in \mathsf{Q}$. Then, by Corollary 5.8, since $\mathfrak{B}_{2[,01]}$ has no proper subalgebra, there is some surjective $h \in \hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) \neq \varnothing$, in which case, by Lemma 2.3, as $(\operatorname{img} h) = \Delta_2$ is not a singleton, $A^2 \neq (\ker h) = h_2^{-1}[\Delta_{B_2}] \in \mathrm{Co}_{\mathsf{Q}}(\mathfrak{A}) \subseteq \{A^2, \Delta_A\}$, and so h is injective, as required, in view of Corollary 5.5. □

Theorem 5.20. *Any relatively semi-simple (more specifically, implicative) quasi-variety* $\mathsf{Q} \subseteq [\mathsf{B}]\mathsf{SMSL}$ *is a sub-variety of* $[\mathsf{B}]\mathsf{DML}$*, in which case it is restricted* $\mho^{\overline{\varphi}}_{V_1|\Omega,wp(\Omega)}$*-implicative, and so "{relatively} ⟨finitely-⟩semi-simple"/"⌈restricted* $\lfloor\mho^{\overline{\varphi}}_{V_1\|\Omega,wp(\Omega)}$*-⌋⌉implicative sub-{quasi-}varieties of* $[\mathsf{B}]\mathsf{SMSL}$ *are exactly sub-varieties of* $[\mathsf{B}]\mathsf{DML}$*.*

Proof. In that case, by Corollary 2.5, Q is generated by $\mathsf{K} \triangleq \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q})$, and so, by Lemmas 5.7 and 5.19, $\mathsf{Q} \subseteq [\mathsf{B}]\mathsf{DML}$. Consider the following complementary cases:

- $\mathsf{K} = \varnothing$,
  in which case $\mathsf{Q} = \mathbf{I}\mathfrak{B}^0_{2[,01]}$.
- $\mathsf{K} \neq \varnothing$.
  Consider the following complementary subcases:
  - $\mathsf{K} \subseteq [\mathsf{B}]\mathsf{NISMSL}$,
    in which case, by Corollary 2.4 and Lemma 5.19, $\mathsf{K} = \mathbf{I}\mathfrak{B}_{2[,01]}$, and so, by Corollary 5.5, $\mathsf{Q} = [\mathsf{B}]\mathsf{BL}$.
  - $\mathsf{K} \nsubseteq [\mathsf{B}]\mathsf{NISMSL}$.
    Consider the following complementary subcases:
    * $\mathsf{K} \subseteq ([\mathsf{B}]\mathsf{SKSL} \cup [\mathsf{B}]\mathsf{NISMSL})$,
      in which case $(\mathsf{K} \setminus [\mathsf{B}]\mathsf{NISMSL}) \subseteq [\mathsf{B}]\mathsf{KL}$, and so, by Lemma 5.19,

$\mathsf{Q} \subseteq [\mathsf{B}]\mathsf{KL}$. Conversely, take any $\mathfrak{A} \in (\mathsf{K} \setminus [\mathsf{B}]\mathsf{NISMSL}) \neq \varnothing$, in which case $(\mathfrak{A}[\restriction\Sigma_{+}^{-}]) \in (\mathsf{SKSL} \setminus \mathsf{NISMSL})$, and so, by Corollary 5.9, there is an embedding e of $\mathfrak{K}_3$ into $\mathfrak{A}[\restriction\Sigma_{+}^{-}]$. Then, [as $a \triangleq e(\langle 0,1\rangle) = \neg^{\mathfrak{A}} a$, by (5.9), $\{\langle 0,0,\bot^{\mathfrak{A}}\rangle, \langle 0,1,a\rangle, \langle 1,1,\top^{\mathfrak{A}}\rangle\}$ is an embedding of $\mathfrak{K}_{3,01}$ into $\mathfrak{A}$, in which case] $\mathfrak{K}_{3[,01]} \in \mathsf{Q}$, and so, by Corollary 5.5, $\mathsf{Q} = [\mathsf{B}]\mathsf{KL}$.

∗ $\mathsf{K} \nsubseteq ([\mathsf{B}]\mathsf{SKSL} \cup [\mathsf{B}]\mathsf{NISMSL}$.
Take any $\mathfrak{B} \in (\mathsf{K} \setminus ([\mathsf{B}]\mathsf{SKSL} \cup [\mathsf{B}]\mathsf{NISMSL})) \neq \varnothing$, in which case, by Corollaries 5.10 and 5.11, there is an embedding f of $\mathfrak{DM}_4$ into $\mathfrak{B}[\restriction\Sigma_{+}^{-}]$, and so $\mathfrak{DM}_4 \in \mathsf{Q}$ in the []-non-optional case. [By contradiction, prove that $\mathfrak{DM}_{4,01} \in \mathsf{Q}$. For suppose $\mathfrak{DM}_{4,01} \notin \mathsf{Q}$, in which case it is not embeddable into $\mathfrak{B}$, and so, by (5.9), both $f(\langle 0|1,0|1\rangle) \neq (\bot|\top)^{\mathfrak{A}}$. Then, by (5.9), $g \triangleq (((\pi_0\restriction(2^2 \times \{\langle 0,1\rangle\}))\circ f)\cup\{\langle\langle 0,0\rangle,\langle 0,0\rangle,\bot^{\mathfrak{B}}\rangle,\langle\langle 1,1\rangle,\langle 1,1\rangle,\top^{\mathfrak{B}}\rangle\})$ is an embedding of $\mathfrak{DM}_6 \triangleq ((\mathfrak{DM}_{4,01} \times \mathfrak{K}_{3,01})\restriction(\operatorname{dom} g))$ into $\mathfrak{B}$, while $\Delta_{K_3}\times\Delta_{K_3}$ is that of $\mathfrak{K}_{3,01}$ into $\mathfrak{DM}_6$, whereas both $\pi_{0\|1}[DM_6] = (DM\|K)_{4\|3}$, in which case $\{\mathfrak{DM}_6, \mathfrak{K}_{3,01}\} \subseteq \mathsf{Q} \not\ni \mathfrak{DM}_{4,01}$, and so, by the Homomorphism Theorem, Lemma 2.3 and the subdirect filtrality of BDML with its simple members $\mathfrak{DM}_{4,01}$ and $\mathfrak{K}_{3,01}$, being due to Corollary 5.5, since $\mathrm{Fi}(2) = \{\wp(N,2) \mid N \in \wp(2)\}$, $\mathrm{Co}_{\{\mathsf{Q}\}}(\mathfrak{DM}_6) = (\{\ker(\pi_i\restriction DM_6) \mid i \in (2\{\setminus 1\})\} \cup \{\Delta_{DM_6}, DM_6^2\}\})$. In this way, since $\Delta_{DM_6} \subsetneq \ker(\pi_1\restriction DM_6) \subsetneq DM_6^2$, for $6 \neq 3 \neq 1$, $\mathfrak{DM}_6 \in (\mathrm{SI}_{\mathsf{Q}}(\mathsf{Q}) \setminus \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}))$, contrary to the relative semi-simplicity of Q.] Thus, by Corollary 5.5, $\mathsf{Q} = [\mathsf{B}]\mathsf{DML}$.

This, by Corollary 5.5 (and Lemma 4.9), completes the argument. □

6. Conclusions

Perhaps, the main problem remained still open whether any ($\mho$-)implicative [quasi-]variety has EDP[R]C scheme ($\mho$). Likewise, the issue whether the stipulations “finitely|finitely-”/“locally-finite” in the formulation of Corollary 4.20/4.21 are necessary remains open as well. Finally, an interesting (though a purely methodological) point remained open is an equational derivation of Corollary 5.5(ii).

References

1. R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia (Missouri), 1974.
2. T.S. Blyth and J.C. Varlet, *On a common abstraction of De Morgan algebras and Stone algebras*, Proc. Roy. Soc. Edinburg A **94** (1983), 301–308.
3. A. I. Budkin and V. A. Gorbunov, *Implicative classes of algebras*, Algebra and Logic **12** (1973), 139–140.
4. T. Frayne, A.C. Morel, and D.S. Scott, *Reduced direct products*, Fundamenta Mathematicae **51** (1962), 195–228.
5. E. Fried, G. Grätzer, and R. Quackenbush, *Uniform congruence schemes*, Algebra Universalis **10** (1980), 176–189.
6. G. Grätzer and E.T. Schmidt, *Ideals and congruence relations in lattices*, Acta. Math. Acad. Sci. Hungar. **9** (1958), 137–175.
7. B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. **21** (1967), 110–121.
8. A. I. Kogalovskiĭ, *On Birkgoff's Theorem*, Uspehi Mat. Nauk **20** (1965), 206–207, In Russian.
9. H. Lakser, *Principal congruences of pdeudocomplemented distributive lattices*, Proceedings of the American Mathematical Society **37** (1973), 32–37.
10. A. I. Mal'cev, *To a general theory of algebraic systems*, Mathematical Collection (New Seria) **35 (77)** (1954), 3–20, In Russian.
11. ———, *Algebraic systems*, Springer Verlag, New York, 1965.

12. A. F. Pixley, *Distributivity and permutability of congruence relations in equational classes of algebras*, Proceedings of the American Mathematical Society **14** (1963), no. 1, 105–109.
13. A. P. Pynko, *Functional completeness and axiomatizability within Belnap's four-valued logic and its expansions*, Journal of Applied Non-Classical Logics **9** (1999), no. 1/2, 61–105, Special Issue on Multi-Valued Logics.
14. ______, *Implicational classes of De Morgan lattices*, Discrete mathematics **205** (1999), 171–181.
15. ______, *Subprevarieties versus extensions. Application to the logic of paradox*, Journal of Symbolic Logic **65** (2000), no. 2, 756–766.
16. ______, *Sequential calculi for many-valued logics with equality determinant*, Bulletin of the Section of Logic **33** (2004), no. 1, 23–32.
17. ______, *A relative interpolation theorem for infinitary universal Horn logic and its applications*, Archive for Mathematical Logic **45** (2006), 267–305.
18. ______, *Subquasivarieties of implicative locally-finite quasivarieties*, Mathematical Logic Quarterly **56** (2010), no. 6, 643–658.
19. H.P. Sankappanavar, *A characterization of principal congruences of De Morgan algebras and its application*, Proceedings of IV Latin American Symposium on Mathematical Logic, Santiago, 1978 (Amsterdam) (A.I. Arruda, R. Chuaqui, and N.C.A. da Costa, eds.), North-Holland Publishing Company, 1980, pp. 341–349.
20. L. A. Skornyakov (ed.), *General algebra*, vol. 2, Nauka, Moscow, 1991, In Russian.
21. D. Ševčovič, *Free non-distributive Morgan-Stone algebras*, New Zealand Journal of Mathematics **94** (1996), 85–94.

Department of Digital Automata Theory (100), V.M. Glushkov Institute of Cybernetics, Glushkov prosp. 40, Kiev, 03680, Ukraine

Email address: `pynko@i.ua`