Compatibility, compossibility, and epistemic modality

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Abstract

We give a theory of epistemic modals in the framework of possibility semantics and axiomatize the corresponding logic, arguing that it aptly characterizes the ways in which reasoning with epistemic modals does, and does not, diverge from classical modal logic.

1 Compatibility versus compossibility

The epistemic modals 'might' and 'must' have intriguing logical properties that fit poorly within standard approaches to modal logic. Most prominently, conjunctions of the form $\varphi \wedge \Diamond\neg\varphi$ or $\Diamond\neg\varphi \wedge \varphi$, dubbed *Wittgenstein sentences* after [24], appear to be contradictory. Evidence for this comes from the incoherence of sentences that embed Wittgenstein sentences, like (1)-(3):

(1) #Suppose John will win but he might not win. [25]

(2) #Either it's raining but it might not be, or it's not raining but it might be. [19]

(3) #Everyone who is a winner might be a loser. [10, 1, 26]

To bring out the significance of these patterns, compare (1)-(3) with minimal variants that embed Moore sentences [21] instead of Wittgenstein sentences:

(4) Suppose John will win but I don't know it.

(5) Either it's raining and I don't know it, or it's not raining and I don't know it.

(6) Everyone who is a winner is, for all I know, a loser.

These are, strikingly, coherent, unlike (1)–(3). While both Moore sentences and Wittgenstein sentences are unassertable, such contrasts in embedding behavior suggest that while Moore sentences are only pragmatically defective, Wittgenstein sentences are semantically defective.

The simplest way to diagnose this defectiveness is to say that Wittgenstein sentences are contradictions, that is, to adopt:

Wittgenstein's Law: $\varphi \wedge \Diamond\neg\varphi \vDash \bot$ and $\Diamond\neg\varphi \wedge \varphi \vDash \bot$.

However, in classical logic, Wittgenstein's Law leads to immediate trouble. If $\varphi \wedge \Diamond\neg\varphi \vDash \bot$, then it follows in classical logic that $\Diamond\neg\varphi \vDash \neg\varphi$ by the classical *Pseudocomplementation* rule:

Pseudocomplementation: If $\varphi \wedge \psi \vDash \bot$, then $\psi \vDash \neg\varphi$.

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But, of course, $\Diamond\neg\varphi$ doesn't entail $\neg\varphi$: 'It might not be raining' doesn't entail 'It's not raining'.

We develop an approach to epistemic modals that adopts Wittgenstein's Law and (hence) rejects Pseudocomplementation. We do this by rejecting the semantic assumption which explains why Pseudocomplementation is classically valid: namely, that *compatibility entails compossibility*. In the classical possible worlds semantics for modal logic, whenever φ and ψ are compatible, i.e., neither entails the negation of the other, it follows that they are compossible, i.e., there is some world where they are both true and hence where $\varphi \wedge \psi$ is true. So, conversely, if $\varphi \wedge \psi$ is never true, it follows that φ and ψ are incompatible, and so ψ entails $\neg\varphi$.

However, we can block this reasoning, for we *need not* accept that compatibility entails compossibility. These are conceptually different notions: two sentences are compossible when they can both be true together; two sentences are compatible when neither entails the negation of the other. And, indeed, these notion can come apart in non-classical versions of a prominent alternative semantic framework for modal logic, namely *possibility semantics* [15, 12]. In possibility semantics, compatibility need not entail compossibility; and so, conversely, incompossibility need not entail incompatibility. The reason for this is that possibilities are *partial*, so that from φ not being true at a possibility, it does not follow that $\neg\varphi$ is true there. Instead, negation is interpreted using an independently specified relation of compatibility. So working with possibilities lets us develop a system where φ and $\Diamond\neg\varphi$ are incompossible—so that Wittgenstein's Law holds—while *also* maintaining that they are compatible—that is, that there are compatible possibilities x and y with φ true at x and $\Diamond\neg\varphi$ true at y, so that $\Diamond\neg\varphi \nvDash \neg\varphi$.

2 Non-classicality (and its limits)

Before developing our system, we lay out some further desiderata. Pseudocomplementation is not the only classical rule that fails for epistemic modals, and we also want to capture some other failures. As [19] observes, *Distributivity* also appears to fail:

Distributivity: $\varphi \wedge (\psi \vee \chi) \dashv\vDash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$.

For if Distributivity were valid, then (7) would entail (2).

(7) It's raining or it's not raining, and it might be raining and it might not be raining.

But that is intuitively wrong: (7) is a coherent statement of ignorance, while (2) is incoherent.

Likewise, *Disjunctive Syllogism* appears to fail for epistemic modals [17]:

Disjunctive Syllogism: $(\varphi \vee \psi) \wedge \neg\psi \vDash \varphi$.

According to Disjunctive Syllogism, the entailment in (8) is valid, but that is intuitively wrong: the premise is just a statement of ignorance, while the conclusion is a claim about the world.

(8) Either Otto is inside or he must be outside, and he might be inside. So, Otto is inside.

We will take these apparent logical facts at face value by giving a theory on which Wittgenstein's Law holds and (hence) Pseudocomplementation fails (so that $\Diamond\neg\varphi \nvDash \neg\varphi$), and Distributivity and Disjunctive Syllogism likewise fail. But, importantly, we aim to preserve as much as possible of the rest of classical modal logic. This last point is worth noting since some attempts to account for the non-classicality of epistemic modals lead to forms of non-classicality for which we think there is not clear evidence, as we discuss in § 6 below.

3 Possibilities

'Might' intuitively expresses *partiality of information*. While there are of course many ways to make sense of this in a framework whose indices are complete possible worlds, it is natural to model partiality of information with models whose indices themselves are partial: that is, which may leave unsettled the truth-value of some sentences. This is not a new idea: in particular, pioneering work in [22], and more recently [11, 2, 7], has given semantics for epistemic modals in terms of partial states. We will develop this idea in a different way, in a non-classical version (building on [8]) of *possibility semantics* for modal logic [15, 12]; in § 6 we will briefly compare this approach to other approaches that also build on partiality. (Possibilities are importantly different from truthmakers [6] but are arguably related to situations [18, 4].)

In possibility semantics, negation is definable in terms of a binary relation of *compatibility*: $\neg\varphi$ is true at possibility x just in case φ is not true at any possibility y compatible with x. This makes possibility semantics partial: it may be that φ is not true at x but is true at some y compatible with x, in which case $\neg\varphi$ is not true at x either. And this lets us draw a distinction that is impossible to draw in possible worlds semantics: we can say that two sentences φ and ψ are *incompossible*, in the sense that there are no possibilities where they are both true, while also maintaining that they are *compatible*, in the sense that there are compatible possibilities x and y with φ true at x and ψ true at y. Thus from $\varphi \wedge \psi \vDash \bot$ it does *not* follow that $\psi \vDash \neg\varphi$. We maintain that this is exactly the situation with φ and $\Diamond\neg\varphi$: these are never true together, but $\Diamond\neg\varphi \nvDash \neg\varphi$.

In more detail, we extend the possibility semantics for a propositional language from [8] to a modal language as follows. A *modal model* is a tuple $\mathcal{M} = \langle S, \between, V, R\rangle$ where S is a nonempty set (of possibilities); $\between$ is a reflexive, symmetric binary relation of *compatibility* on S; V is an atomic valuation, taking any sentence letter to a subset of S;[1] and R is a reflexive binary accessibility relation, representing epistemic possibility.[2] Given a modal model and possibility $x \in S$, we interpret our modal language with $[\![\cdot]\!]^x$, defined as follows:

- $[\![p]\!]^x = 1$ iff $x \in V(p)$, for sentence letters p;
- $[\![\varphi \wedge \psi]\!]^x = 1$ iff $[\![\varphi]\!]^x = 1$ and $[\![\psi]\!]^x = 1$;
- $[\![\Box\varphi]\!]^x = 1$ iff $\forall y \in R(x)\ [\![\varphi]\!]^y = 1$.

So far, $[\![\cdot]\!]^x$ is just as in possible worlds semantics for modal logic. But, crucially, we interpret negation using *incompatibility* rather than *lack of truth*:

- $[\![\neg\varphi]\!]^x = 1$ iff $\forall y \between x\ [\![\varphi]\!]^y \neq 1$.

Finally, we use De Morgan's law to define $\varphi \vee \psi$ as $\neg(\neg\varphi \wedge \neg\psi)$ and Duality to define $\Diamond\varphi$ as $\neg\Box\neg\varphi$, which yields:

- $[\![\varphi \vee \psi]\!]^x = 1$ iff $\forall y \between x\ \exists z \between y : [\![\varphi]\!]^z = 1$ or $[\![\psi]\!]^z = 1$.
- $[\![\Diamond\varphi]\!]^x = 1$ iff $\forall y \between x\ \exists z \in R(y)\ \exists u \between z : [\![\varphi]\!]^u = 1$.

[1] The subsets must meet a closure condition ensuring that propositions are well-behaved, or technically *regular*, where $A \subseteq S$ is regular iff $\forall x \in S$, if $x \notin A$, then $\exists y \between x\ \forall z \between y\ z \notin A$.

[2] R must also satisfy an interaction condition with $\between$ to ensure that modal sentences correspond to regular sets, namely: if $\exists z : xRz$ and $y \between z$, then $\exists x' \between x\ \forall x'' \between x'\ \exists z : x''Rz$ and $y \between z$.

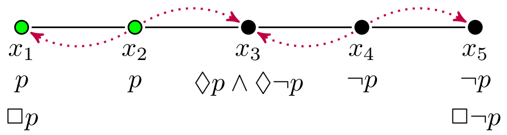

Figure 1: An epistemic model for a language with a sentence letter p; possibilities where p is true are in green. Solid lines represent $\between$ (compatibility), while dotted lines represent R (epistemic accessibility). Compatibility and epistemic accessibility are reflexive (reflexive arrows are omitted) but need not be transitive. Note the failure of Pseudocomplementation: at no possibility is $p \wedge \Diamond\neg p$ true, but $\Diamond\neg p$ does not entail $\neg p$, as the former is true at the middle possibility while the latter is not (neither p nor $\neg p$ is true there). Note also the failure of Distribution: $(p \vee \neg p) \wedge (\Diamond p \wedge \Diamond\neg p)$ is true at the middle possibility (as $p \vee \neg p$ is true at every possibility), while $(p \wedge \Diamond\neg p) \vee (\neg p \wedge \Diamond\neg p)$ is not true at any possibility.

The key move to enforce Wittgenstein's Law is to add a constraint stating that for any possibility x, there is a possibility y such that everything true at x *must* be true at y:

- *Knowability*: for all $x \in S$, there is a $y \in S$ such that for all $z \in R(y)$, $z \sqsubseteq x$,

where $z \sqsubseteq x$ ("z refines x") means that every possibility compatible with z is also compatible with x. That $z \sqsubseteq x$ ensures that every proposition true at x is also true at z. Hence *Knowability* says that every proposition true at x is true at all possibilities epistemically accessible from y and hence *must* be true at y. The intuition is that possibilities represent states that could be the information of some agent. So any possibility has to be "known" to obtain somewhere in modal space. Call a modal model that satisfies *Knowability* an *epistemic model*.

4 Consequence

We define consequence as usual: $\varphi \vDash \psi$ if at each point in each epistemic model, if φ is true, then ψ is true. This consequence relation has exactly the profile we set out to obtain.

First, Wittgenstein's Law is valid. Consider $\varphi \wedge \Diamond\neg\varphi$ (or $\Diamond\neg\varphi \wedge \varphi$, as the reasoning is the same). Suppose for contradiction this is true at x. By *Knowability*, there is a possibility y where this *must* be true, i.e., $[\![\Box(\varphi \wedge \Diamond\neg\varphi)]\!]^y = 1$. Hence by Distribution of $\Box$ over conjunction, $[\![\Box\varphi \wedge \Box\Diamond\neg\varphi]\!]^y = 1$. Then by Factivity for $\Box$, $[\![\Box\varphi \wedge \Diamond\neg\varphi]\!]^y = 1$. But by Duality and Double Negation Elimination (which is valid according to the semantics), $\Box\varphi \wedge \Diamond\neg\varphi$ is equivalent to the contradiction $\Box\varphi \wedge \neg\Box\varphi$ and so is never true. So Wittgenstein sentences are contradictory, since they are not "knowable" and so are not true at any possibility.

Importantly, not only are Wittgenstein sentences contradictions, but also they are *everywhere substitutable for contradictions*, meaning we accommodate incoherence data like (1)–(3) (once our system is extended with any reasonable treatment of attitudes and quantifiers). This is important to note, because in some systems Wittgenstein sentences are contradictions but cannot always be substituted for contradictions, as in domain semantics [25], which thus cannot account for the incoherence of disjunctive or quantified Wittgenstein sentences like (2) and (3).

Second, Pseudocomplementation fails, as desired: $\varphi \wedge \Diamond\neg\varphi$ is never true, but $\Diamond\neg\varphi \nvDash \neg\varphi$. This is because $\Diamond\neg\varphi$ and φ are still *compatible*, even though they are not compossible: there are possibilities x, y with $x \between y$ and $[\![\varphi]\!]^x = [\![\Diamond\neg\varphi]\!]^y = 1$, as in x_2 and x_3 in Figure 1 with p as φ. Such models show that $\Diamond\neg\varphi \nvDash \neg\varphi$, since $[\![\Diamond\neg\varphi]\!]^y = 1$ while $[\![\neg\varphi]\!]^y \neq 1$.

Finally, Distributivity and Disjunctive Syllogism are not valid. First, $(\varphi \vee \neg\varphi) \wedge (\Diamond\varphi \wedge \Diamond\neg\varphi)$ is consistent and equivalent to $\Diamond\varphi \wedge \Diamond\neg\varphi$, as desired. Contrary to Distributivity, it does *not* entail $(\varphi \wedge \Diamond\neg\varphi) \vee (\neg\varphi \wedge \Diamond\varphi)$, which, by contrast, is a disjunction of contradictions and hence is contradictory. Second, $\varphi \vee \Box\neg\varphi$ is a tautology, so $(\varphi \vee \Box\neg\varphi) \wedge \Diamond\varphi$ is equivalent to $\Diamond\varphi$ and hence does not entail φ, contrary to Disjunctive Syllogism.

5 Logic

Principles 1-15 in Figure 2 constitute a logic that is sound and complete with respect to the possibility semantics of § 3. This logic extends *orthologic* (a weakening of classical logic that lacks Pseudocomplementation, Distributivity, and Disjunctive Syllogism [8]) with modal operators satisfying the logical analogue of *Knowability*, namely that if $\Box\varphi \vdash \bot$, then $\varphi \vdash \bot$.

It is worth noting that while this logic accounts for the non-classicality introduced by epistemic modals, it still preserves much of classical logic. Excluded Middle, Non-Contradiction, and De Morgan's laws remain valid. Even though it may be that neither φ nor $\neg\varphi$ is true at x, their disjunction will always be, given our clause for disjunction. $\varphi \wedge \neg\varphi$ cannot be true at any possibility since compatibility is reflexive. De Morgan's laws hold in virtue of our definition of disjunction and classical properties of our treatment of negation.

By extending our language with special Boolean propositional variables, standing for non-modal propositions, we can capture even more of classical logic. In [14], we further constrain our epistemic models so that we validate all classical reasoning in the Boolean fragment of our language (the closure of the Boolean propositional variables under $\wedge$ and $\neg$), as well as in fragments of the language that contain modals but do not mix different levels of modal embedding. Define *epistemic levels* inductively: the first epistemic level is the closure of the set of Boolean variables under $\wedge$ and $\neg$. For any n, construct the $n+1^{st}$ epistemic level by taking the sentences with the form $\Box\varphi$, for φ in the n^{th} epistemic level, and closing under $\wedge$ and $\neg$. All the non-classicality introduced by epistemic modals comes from sentences that do not belong to any epistemic level (for instance, $\varphi \wedge \Diamond\neg\varphi$ conjoins a sentence φ of epistemic level n with a sentence $\Diamond\neg\varphi$ of epistemic level $n+1$). To capture this insight, we can further constrain epistemic models so that if φ and ψ are in the same epistemic level and are compatible in the model, then they are compossible in the model. Then the set of sentences at epistemic level n has a fully classical logic, as captured by principle 16 in Figure 2.

1. $\varphi \vdash \top$;	6. $\neg\neg\varphi \vdash \varphi$;
2. $\varphi \vdash \varphi$;	7. $\varphi \wedge \neg\varphi \vdash \psi$;
3. $\varphi \wedge \psi \vdash \varphi$;	8. if $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;
4. $\varphi \wedge \psi \vdash \psi$;	9. if $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;
5. $\varphi \vdash \neg\neg\varphi$;	10. if $\varphi \vdash \psi$, then $\neg\psi \vdash \neg\varphi$.
11. if $\varphi \vdash \psi$, then $\Box\varphi \vdash \Box\psi$;	14. $\Box\varphi \vdash \varphi$;
12. $\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$;	15. if $\Box\varphi \vdash \bot$, then $\varphi \vdash \bot$;
13. $\varphi \vdash \Box\top$;	16. $\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ provided α, β, γ are in the same epistemic level.

Figure 2: Principles 1-15 constitute the logic EO of *epistemic orthologic*, corresponding to epistemic models. Principles 1-10 characterize orthologics. Principle 15 is the proof-theoretic analogue of *Knowability*. Adding principle 16, corresponding to the extra constraint on epistemic models concerning epistemic levels in the main text, strengthens the logic from EO to EO^+.

6 Comparisons

While we do not have space for complete comparisons with other systems, we will briefly explain ways in which we think the approach outlined above improves on existing theories. The first point concerns non-classicality. While we can make sense of the failure of classical laws that indeed appear to fail for epistemic modals, we still preserve much of classical logic. This distinguishes our system from prominent systems like dynamic semantics and data semantics, where Excluded Middle fails [10, 22], and from state-based systems where De Morgan's laws fail [11].

A second point is methodological. Following the call in [13], we have given a sound and complete axiomatization of the validities of our system, thus directly characterizing the logical inferences that our system predicts will strike speakers as valid. Since capturing felt (in)validity is a central part of the task of natural language semantics, having such a characterization is an essential part of any adequate theory of a fragment of natural language.

The third point concerns empirical adequacy vis-à-vis Wittgenstein sentences. Unlike in dynamic semantics [23, 10], our account is order insensitive, capturing the infelicity of sentences that embed either $\varphi \wedge \Diamond\neg\varphi$ or $\Diamond\neg\varphi \wedge \varphi$. It is also fully general over substitution instances of φ; that is, it predicts Wittgenstein sentences are contradictions for any φ, unlike dynamic semantics, where this need not hold when φ is modal [20]. And Wittgenstein sentences are, again, not just contradictions but also everywhere substitutable for contradictions, giving our account considerably more empirical viability than systems like [25, 16, 3] where they are contradictions but not everywhere substitutable for contradictions.

Our final point starts with the observation that we can naturally form judgments about the probabilities of sentences containing modals and conditionals. In the full paper [14], we show how to add measures to possibility models to yield these judgments. Here we only make a conceptual point. Possibilities are (incomplete) ways for the world to be, and so it makes sense to specify measures over them. By contrast, in many treatments of epistemic modals that are likewise built around some notion of partiality, indices are treated as *information states* of some kind, and it is unclear how to construct measures over such things in a reasonable way (see [5, 9] for discussion). This problem is more tractable for some informational semantics than others. For instance, [9] shows how to extend probability measures to algebras of sets of pairs of a world and a set of possible worlds, which are the indices of domain semantics [25]. However, domain semantics is empirically untenable for the reasons noted above. The state-based semantics of [11] is empirically much more successful, but it faces a deep problem with probabilities, namely, that it yields $p \vDash \Box p$. This implies that any measure on an algebra of sets of states will always assign at least as great a probability to $\Box p$ as to p, but this is unrealistic: there is .5 probability that this fair coin will land heads but very little probability that it *must* land heads. By contrast, this problem does not arise for us, since we have $p \nvDash \Box p$.

7 Conclusion

To capture the way in which 'might' expresses partiality of information, we have developed a theory of epistemic modals built on possibility semantics, where indices are themselves partial. This lets us disentangle the notions of *compatibility* and *compossibility* that are run together in possible worlds semantics. Once we distinguish these, we can validate Wittgenstein's Law without arriving at the absurd conclusion that $\Diamond\neg\varphi \vDash \neg\varphi$. This approach captures the ways in which epistemic modals diverge from classical logic without introducing excessive non-classicality. In the full paper [14], we also give an algebraic characterization of our logic, equip our models with measures to capture probability judgments, and explore how to integrate conditionals.

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