Near-Square Primes Conjecture

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Abstract. In 1912, Edmund Landau listed four basic problems about prime numbers in the International Congress of Mathematicians. These problems are now known as Landau's problems. Landau's fourth problem asked whether there are infinitely many primes which are of the form $n^2 + 1$ for some integer n. This problem remains open and it is known as the Near-square primes conjecture. We prove this conjecture is indeed true.

1. Results

Definition 1.1. Given a function $f : \mathbb{N} \to \mathbb{R}$, we define

$$\lim_{n \to \infty} I(f(n)) = 1$$

when $\exists m_0 \in \mathbb{N}$ such that $\forall n > m_0 : f(n) \in \mathbb{Z}$ and

$$\lim_{n \to \infty} I(f(n)) = 0$$

when $\nexists m_0 \in \mathbb{N}$ such that $\forall n > m_0 : f(n) \in \mathbb{Z}$.

Lemma 1.2. *Given a function $f : \mathbb{N} \to \mathbb{R}$ and an irrational number α, we have that*

$$\lim_{n \to \infty} I(\alpha \times f(n)) = 0$$

when

$$\lim_{n \to \infty} I(f(n)) = 1.$$

Proof. Certainly, a number $\alpha \times k$ is not an integer when k is an integer even though k could be no matter how large we want. □

Theorem 1.3. *There are infinitely many primes which are of the form $n^2 + 1$ for some integer n.*

Proof. Suppose, there are not infinitely many primes which are of the form n^2+1 for some integer n. In number theory, Wilson's theorem states that a natural number $n > 4$ is a composite number if and only if the product of all the positive integers less than n is multiple of n [2]. That is the factorial $(n-1)! = 1 \times 2 \times 3 \times \cdots \times (n-1)$ satisfies

$$(n-1)! \equiv 0 \pmod{n}$$

exactly when n is a composite number [2]. In this way, if the Near-square primes conjecture is false, then we would have that $n^2 + 1$ must be a composite number when n tends to infinity. Consequently, we obtain that

$$\lim_{n \to \infty} I(\frac{n^2!}{n^2+1}) = 1.$$

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We know that

$$\prod_{j=1}^{\infty} \frac{(p_j^2 - 1)}{(p_j^2 - 1)} = 1$$

where p_j is the j^{th} prime number. We also know that

$$\lim_{n\to\infty} I(\frac{n^2!}{n^2+1} \times 1) = 1$$

and thus, we obtain that

$$\lim_{n\to\infty} I(\frac{n^2!}{n^2+1} \times \prod_{j=1}^{\infty} \frac{(p_j^2 - 1)}{(p_j^2 - 1)}) = 1.$$

Since $n^2 + 1$ should be a composite number when n tends to infinity, then this must be in the form of $p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_m^{a_m}$ such that $p_1, p_2, \cdots, p_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are positive integers according to the Fundamental theorem of arithmetic [2]. In the case of $2 \notin \{a_1, a_2, \cdots, a_m\}$, then we can pick some $a_i \in \{a_1, a_2, \cdots, a_m\}$ such that $a_i > 3$ or $a_i = 1$ and obtain that

$$\lim_{n\to\infty} I(\frac{n^2!}{p_i}) = 1$$

and

$$\lim_{n\to\infty} I(\frac{n^2!}{p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_i^{a_i-1} \times \cdots \times p_m^{a_m}}) = 1$$

where $n^2 + 1 = p_i \times p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_i^{a_i-1} \times \cdots \times p_m^{a_m}$, $a_i - 1 > 2$ or $a_i - 1 = 0$ and there is no square of a prime number p_j^2 that is eliminated from the division $\frac{n^2!}{n^2+1}$ in the numerator $n^2!$. Certainly, we will only eliminate from the numerator $n^2!$, the numbers $p_i \leq n^2$ and $p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_i^{a_i-1} \times \cdots \times p_m^{a_m} \leq n^2$ such that $p_i \neq p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_i^{a_i-1} \times \cdots \times p_m^{a_m}$, where p_i and $p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_i^{a_i-1} \times \cdots \times p_m^{a_m}$ are not square numbers. We are always able to obtain such exponent $a_i > 3$ when $\forall a_j \in \{a_1, a_2, \cdots, a_m\} : a_j > 2$, due to the number $n^2 + 1$ is not in the form of x^3 for some natural number x when n tends to infinity. Certainly, according to the Catalan's conjecture, the only solution in the natural numbers of

$$x^a - y^b = 1$$

for $a, b > 1$, $x, y > 0$ is $x = 3$, $a = 2$, $y = 2$, $b = 3$ [1]. In addition, we are always able to obtain such exponent $a_i = 1$ when $\exists a_j \in \{a_1, a_2, \cdots, a_m\} : a_j = 1$. In the case of $2 \in \{a_1, a_2, \cdots, a_m\}$, then we can pick those prime divisors $p_{k_1}, p_{k_2}, \cdots, p_{k_t}$ which have 2 as exponent in $n^2 + 1$ and obtain that

$$\lim_{n\to\infty} I(\frac{n^2!}{(p_{k_1} \times p_{k_2} \times \cdots \times p_{k_t})}) = 1$$

and

$$\lim_{n\to\infty} I(\frac{n^2!}{(p_{k_1} \times p_{k_2} \times \cdots \times p_{k_t}) \times p_{b_1}^{r_1} \times p_{b_2}^{r_2} \times \cdots \times p_{b_s}^{r_s}}) = 1$$

where $n^2 + 1 = (p_{k_1} \times p_{k_2} \times \cdots \times p_{k_t}) \times (p_{k_1} \times p_{k_2} \times \cdots \times p_{k_t}) \times p_{b_1}^{r_1} \times p_{b_2}^{r_2} \times \cdots \times p_{b_s}^{r_s}$, $2 \notin \{r_1, r_2, \cdots, r_s\}$ and there is no square of a prime number p_j^2 that is eliminated from the division $\frac{n^2!}{n^2+1}$ in the numerator $n^2!$. Certainly, we will only eliminate from the numerator $n^2!$, the numbers $(p_{k_1} \times p_{k_2} \times \cdots \times p_{k_t}) \leq n^2$ and $(p_{k_1} \times p_{k_2} \times \cdots \times p_{k_t}) \times p_{b_1}^{r_1} \times p_{b_2}^{r_2} \times \cdots \times p_{b_s}^{r_s} \leq n^2$ such that $(p_{k_1} \times p_{k_2} \times \cdots \times p_{k_t}) \neq$

$(p_{k_1} \times p_{k_2} \times \cdots \times p_{k_t}) \times p_{b_1}^{r_1} \times p_{b_2}^{r_2} \times \cdots \times p_{b_s}^{r_s}$, where $(p_{k_1} \times p_{k_2} \times \cdots \times p_{k_t})$ and $(p_{k_1} \times p_{k_2} \times \cdots \times p_{k_t}) \times p_{b_1}^{r_1} \times p_{b_2}^{r_2} \times \cdots \times p_{b_s}^{r_s}$ are not square numbers. We are always able to obtain such number $p_{b_1}^{r_1} \times p_{b_2}^{r_2} \times \cdots \times p_{b_s}^{r_s} \neq 1$, due to $n^2 + 1$ is not in the form of x^2 for some natural number x when n tends to infinity. In this way, we have that

$$\lim_{n \to \infty} I(\frac{n^2!}{n^2+1} \times \prod_{j=1}^{\infty} \frac{(p_j^2 - 1)}{(p_j^2 - 1)}) = 1$$

is equivalent to

$$\lim_{n \to \infty} I(\prod_{j=1}^{\infty} (p_j^2) \times g(n) \times \prod_{j=1}^{\infty} \frac{(p_j^2 - 1)}{(p_j^2 - 1)}) = 1$$

where

$$\lim_{n \to \infty} I(g(n)) = 1$$

since there is no square of a prime number p_j^2 that must necessarily be eliminated from the division $\frac{n^2!}{n^2+1}$ in the numerator $n^2!$ within any case. In addition, we can transform this limit into

$$\lim_{n \to \infty} I(\prod_{j=1}^{\infty} \frac{p_j^2}{p_j^2 - 1} \times h(n)) = 1$$

where

$$\lim_{n \to \infty} I(h(n)) = \lim_{n \to \infty} I(g(n) \times \prod_{j=1}^{\infty} (p_j^2 - 1)) = \lim_{n \to \infty} I(g(n) \times \prod_{p_j < n^2+1} (p_j^2 - 1)) = 1$$

since all the prime numbers p_j are lesser than $n^2 + 1$ when n tends to infinity. However,

$$\lim_{n \to \infty} I(\prod_{j=1}^{\infty} \frac{p_j^2}{p_j^2 - 1} \times h(n)) = 1$$

would be the same as

$$\lim_{n \to \infty} I(\frac{\pi^2}{6} \times h(n)) = 1$$

since we have that

$$\prod_{j=1}^{\infty} \frac{p_j^2}{p_j^2 - 1} = \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of the result in the Basel problem [2]. Hence, we obtain a contradiction since

$$\lim_{n \to \infty} I(\frac{\pi^2}{6} \times h(n)) = 0$$

according to the Lemma 1.2. To sum up, we have that our assumption that the Near-square primes conjecture were false is incorrect and therefore, we obtain that the conjecture should be necessarily true. □

References

[1] Preda Mihailescu. Primary cyclotomic units and a proof of catalans conjecture. *Journal für die reine und angewandte Mathematik*, 2004(572):167–195, 2004.

[2] David G. Wells. *Prime Numbers, The Most Mysterious Figures in Math*. John Wiley & Sons, Inc., 2005.

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