Notes on Partial Derivatives Equations and Utility Functions

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ABSTRACT

This note is about the construction of a utility function through the resolution of an economic problem.

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1.Introduction

Utility concept comes from Bernoulli (1738) and has been related to Economics since the very beginning. In the 18th century, Laplace discussed the Bernoulli principle and its relevance to insurance systems. Later, Barrois (1832) presented a theory of fire insurance based on Laplace's work and in the Bernoulli principle. Just in the middle of the 20th century, utility theory reappeared with the von Neumann and Morgenstern's (1947) utility function derived from a set of axioms governing a preference ordering.

Due to Borch (see Borch [1] and also Borch [2]), risk theory has grown beyond ruin theory.

Recently and particularly, utility functions have been applied also in the context of neuroeconomics. The utility theory and the theory of preferences have been applied to study human behavior and to work the decision theory. Often, these theories, working in the behavior theories area, are associated to game theory. For example, neuroscientific research apparently proves the testable proposition of the existence of social preferences in public goods' games (see Neumärker [3], pp. 68). In this paper, a formulation on the area of utility functions is made through mathematical tools, deducting a general function for utility and presenting some possible examples.

2.The Formulation

Consider a Universe with n goods $x_1, x_2, \dots, x_n$ and an individual consumer with income Y. Suppose that this consumer tries to maximize its utility function

$$U = U(x_1, x_2, \dots, x_n).$$

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The mathematical formulation of the problem is, evidently,

$$Max.\, U(x_1, x_2, \dots, x_n)$$

$$s.to{:}\, x_1 P_{x_1} + x_2 P_{x_2} + \cdots + x_n P_{x_n} = Y$$

being P_{x_i} the price of the good $i, i = 1, 2, \dots, n$.

It is a constrained optimization problem. The Lagrange's multipliers method may be applied in its solution (see for instance Ferreira and Amaral [4]). The Lagrange function is

$$L(x_1, x_2, \dots, x_n, \lambda) = U(x_1, x_2, \dots, x_n) + \lambda\left(Y - x_1 P_{x_1} - x_2 P_{x_2} - \cdots - x_n P_{x_n}\right) \qquad (1).$$

The first order conditions are

$$\begin{gathered} \frac{\partial U}{\partial x_1} - \lambda P_{x_1} = 0 \\ \frac{\partial U}{\partial x_2} - \lambda P_{x_2} = 0 \\ \vdots \\ \frac{\partial U}{\partial x_n} - \lambda P_{x_n} = 0 \\ x_1 P_{x_1} + x_2 P_{x_2} + \cdots + x_n P_{x_n} = Y. \end{gathered}$$

Using the first and the last conditions it is obtained, considering $x_1 \neq 0$,

$$\frac{\partial U}{\partial x_1} - \lambda \left(\frac{Y}{x_1} - \frac{x_2}{x_1} P_{x_2} - \cdots - \frac{x_n}{x_1} P_{x_n}\right) = 0$$

or

$$x_1 \frac{\partial U}{\partial x_1} + \lambda x_2 P_{x_2} + \cdots + \lambda x_n P_{x_n} = \lambda Y.$$

As $\lambda P_{x_i} = \frac{\partial U}{\partial x_i}, i = 1, 2, \dots, n$, the former equation becomes

$$x_1 \frac{\partial U}{\partial x_1} + x_2 \frac{\partial U}{\partial x_2} + \cdots + x_n \frac{\partial U}{\partial x_n} = \lambda Y$$

and noting that

$$\lambda = \frac{1}{P_{x_1}} \frac{\partial U}{\partial x_1}$$

it assumes the form

$$\left(x_1 - \frac{Y}{P_{x_1}}\right) \frac{\partial U}{\partial x_1} + x_2 \frac{\partial U}{\partial x_2} + \cdots + x_n \frac{\partial U}{\partial x_n} = 0 \qquad (2).$$

It is a first order homogeneous partial derivatives equation (see Ferreira and Amaral [5]). Solving:

$$U(x_1, x_2, \dots, x_n) = F\left(\frac{Y}{P_{x_1} + P_{x_2} + \cdots + P_{x_n}}\right) \qquad (3).$$

Notes:

- If U_1 and U_2 solve (2), $c_1U_1 + c_2U_2, c_1, c_2 \in \mathbb{R}$ also solve it: $\left(x_1 - \frac{Y}{P_{x_1}}\right)\left(c_1 \frac{\partial U_1}{\partial x_1} + c_2 \frac{\partial U_2}{\partial x_1}\right) + x_2\left(c_1 \frac{\partial U_1}{\partial x_2} + c_2 \frac{\partial U_2}{\partial x_2}\right) + \cdots + x_n\left(c_1 \frac{\partial U_1}{\partial x_n} + c_2 \frac{\partial U_2}{\partial x_n}\right) = c_1\left(\left(x_1 - \frac{Y}{P_{x_1}}\right) \frac{\partial U_1}{\partial x_1} + x_2 \frac{\partial U_1}{\partial x_2} + \cdots + x_n \frac{\partial U_1}{\partial x_n}\right) + c_2\left(\left(x_1 - \frac{Y}{P_{x_1}}\right) \frac{\partial U_2}{\partial x_1} + x_2 \frac{\partial U_2}{\partial x_2} + \cdots + x_n \frac{\partial U_2}{\partial x_n}\right) = c_1 0 + c_2 0 = 0$. So the functions' solutions fulfill the linearity property and, of course, a convex combination of solutions is also a solution.
- $F(.)$ is any differentiable function.
- $Y = x_1P_{x_1} + x_2P_{x_2} + \cdots + x_nP_{x_n}$ is the income constraint.
- It is easy to see that (3) is a solution of (2) substituting directly.
- The expression (3) evidences the utility functional dependency from the whole goods and the income.

3.Some Examples

One concretization of (3) may be

$$U(x_1, x_2, \dots, x_n) = \alpha^{\frac{Y}{P_{x_1} + P_{x_2} + \cdots + P_{x_n}}} \quad (4)$$

for which $U(1, 1, \dots, 1) = \alpha$. That is, α is a standard utility: the value of the utility when unitary quantities of every good are used.

Defining $z_i = \alpha^{x_i}, i = 1,2, \dots, n$, (4) becomes

$$V(z_1, z_2, \dots, z_n) = U(\log_\alpha z_1, \log_\alpha z_2, \dots, \log_\alpha z_n) = {z_1}^{\alpha_1} {z_2}^{\alpha_2} \dots {z_n}^{\alpha_n} \quad (5)$$

where $\alpha_i = \frac{P_{x_i}}{P}, i = 1,2, \dots, n$ and $P = \sum_{i=1}^{n} P_{x_i}$ are the standardized prices of each good. Formally, (5) is a Cobb-Douglas function. That is, in terms of the standard utility base exponential of each quantity, (4) assumes the Cobb-Douglas utility form. Note that $\sum_{i=1}^{n} \alpha_i = 1$.

Another example is

$$U(x_1, x_2, \dots, x_n) = \beta \frac{Y}{P_{x_1} + P_{x_2} + \cdots + P_{x_n}} \quad (6)$$

being now β the standard utility.

The expression (6) may be written as

$$U(x_1, x_2, \dots, x_n) = \beta \sum_{i=1}^{n} \alpha_i x_i \quad (7)$$

with α_i defined above. So, in this case the utility is given by a linear combination of the quantities, which weights are the standardized prices, multiplied by the standard utility.

And, finally, as last example

$$U(x_1, x_2, \dots, x_n) = \gamma \, ln \frac{Y}{P_{x_1} + P_{x_2} + \cdots + P_{x_n}} \quad (8)$$

for which $U(1, 1, \dots, 1) = 0$ and $U(e, e, \dots, e) = \gamma$.

Evidently

$$U(x_1, x_2, \dots, x_n) = ln(\sum_{i=1}^{n} \alpha_i x_i)^\gamma \quad (9).$$

4.Concluding Remarks

After the formulation of an evident economic problem related with the utility of goods, a differential equation with partial derivatives is deduced. A general solution is determined and through its concretization in functional terms some analytical expressions for the utility are presented. They are quite interpretable in terms of the economic reality. One next step, very difficult, is to look for the deducted equations fitness to real data. It is surely an interesting research field.

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