Analyzing Quasi-Photons in a Novel Field Theoretic Framework

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This paper explores the concept of Quasi Photons within the framework of a specific model. The model is defined in terms of a field theory with two sectors: one representing a harmonic oscillator with negative norm and the other capturing the dynamics of a massless scalar field with a negative mass squared in the subspace with negative norm. The analysis focuses on the construction of field operators, their harmonic oscillator representation, and their corresponding path integral formulation. The paper further investigates the real-space expressions for the propagator in the case where the mass parameter is zero. Explicit expressions for the propagator and its various components are derived, shedding light on the behavior of the system in both the positive and negative norm sectors. The exploration emphasizes the implications of this unique model, providing insights into the nature of quasi photons and their characteristics. The results presented in this paper contribute to our understanding of unconventional quantum field theories and open up avenues for further research into the intriguing properties of quasi photons.

I. INTRODUCTION

The realm of quantum field theory continually unravels new facets of the quantum world, challenging our preconceptions and expanding the boundaries of our understanding. In this pursuit, we delve into the intriguing concept of "Quasi Photons," a phenomenon that emerges from a unique model defined within the framework of a field theory. This model involves two distinct sectors, one characterized by a harmonic oscillator with negative norm and the other by the dynamics of a massless scalar field with a negative mass squared in the subspace with negative norm. As we embark on this exploration, it becomes evident that the conventional notions of photons and their behavior undergo a profound transformation within the confines of this unconventional model. The purpose of this paper is to elucidate the intricate details of the "Quasi Photon" phenomenon, offering a comprehensive analysis of the field operators, their harmonic oscillator representations, and the associated path integral formulation. A particular focus is placed on the case where the mass parameter is zero, unraveling the real-space expressions for the propagator. The derived explicit expressions provide a nuanced understanding of the system's dynamics, both in the positive and negative norm sectors. This investigation promises to unveil the distinct characteristics of quasi photons, shedding light on their nature and behavior within the framework of this unconventional quantum field theory.By presenting these results, this paper aims to contribute to the broader discourse on unconventional quantum field theories, inviting further inquiry into the implications of quasi photons and their role in shaping our understanding of the quantum landscape.

II. MODEL

The foundational structure of our quantum field theory model is succinctly described by the following action functional:

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$$\begin{aligned} S_0 &= \int dx \left(-\frac{1}{4} F_{\mu\nu} f(-\Box) F^{\mu\nu} - \frac{\xi}{2} \partial_\alpha A f(-\Box) \partial^\alpha A \right) \\ &= \frac{1}{2} \int dx \left[-\partial_\mu A_\nu f(-\Box) \partial^\mu A^\nu + \partial_\mu A_\nu f(-\Box) \partial^\nu A^\mu + \xi A_\mu f(-\Box) \partial^\mu \partial_\alpha A \right] \\ &= \frac{1}{2} \int dx A_\mu [g^{\mu\nu} \Box - (1-\xi) \partial^\mu \partial^\nu] f(-\Box) A_\nu \\ &= \frac{1}{2} A D_0^{-1} A, \end{aligned} \tag{1}$$

where the Feynman gauge condition is enforced ($\xi = 1$), and the propagator and self-energy tensor are defined as:

$$D_0^{-1}(p) = -g^{\mu\nu} p^2, \tag{2}$$

$$\Pi^{++\mu\nu}(q) = -g^{\mu\nu} \ell^2 (q^2)^2, \tag{3}$$

$$f(p^2) = 1 + \ell^2 (p^2)^2, \tag{4}$$

$$D(k) = \frac{1}{k^2 + i\epsilon} - \frac{1}{k^2 + \frac{1}{\ell^2} - i\epsilon} = \frac{1}{k^2 + i\epsilon} \frac{1}{1 + \ell^2 (k^2 - i\epsilon)} = \frac{1}{k^2 + \ell^2 (k^2)^2 + i\epsilon}. \tag{5}$$

In these expressions, $\Box$ denotes the D'Alembertian operator, and A, $F_{\mu\nu}$ represent the vector potential and field strength tensor, respectively. The parameter ξ governs the gauge-fixing term, while ℓ introduces a characteristic length scale into the theory. The inverse propagator $D_0^{-1}(p)$, self-energy tensor $\Pi^{++\mu\nu}(q)$, and form factor $f(p^2)$ elegantly encapsulate the intricate dynamics of the quasi-photon within this unconventional quantum field theory.

III. FIELD OPERATOR

A. Harmonic Oscillator with Negative Norm

Consider a linear space $H = H_+ \oplus H_-$, where $\eta H_\sigma = \sigma H_\sigma$.

Matrix elements are given by $\langle m|A|n\rangle = (m, \eta A n)$.

The adjoint is denoted as $\langle m|\bar{A}|n\rangle = \langle n|A|m\rangle^*$, where $\bar{A} = \sigma_A A$ and $\sigma = \pm 1$.

Eigenvalue equations are expressed as $A\lambda\rangle = \lambda|\lambda\rangle$ and $A\rho\rangle = \rho|\rho\rangle$, subject to the constraint $(\lambda - \sigma_A \rho^*)\langle\rho|\lambda\rangle = 0$, yielding a real spectrum for skew-adjoint operators with non-orthogonal, degenerate eigenvectors.

The following relations hold: $\bar{q}_\sigma = \sigma q_\sigma$, $\bar{p}_\sigma = \sigma p_\sigma$, and $[q_\sigma, p_\sigma] = i$.

The operators a_σ are defined as $a_\sigma = (m\omega q_\sigma + i p_\sigma)/\sqrt{2m\omega}$, with $q_\sigma = (a_\sigma + \sigma \bar{a}_\sigma)/\sqrt{2m\omega}$, $p_\sigma = (a_\sigma - \sigma \bar{a}_\sigma)/\sqrt{2}i$, and $[a_\sigma, \bar{a}_\sigma] = \sigma$.

The corresponding Hamiltonian H_σ is given by $H_\sigma = \sigma\omega(\bar{a}_\sigma a_\sigma + \frac{1}{2})$, where $b = a_+$ or $\bar{a}_-$, $\bar{b} = \bar{a}_+$ or a_-, and $[b, \bar{b}] = 1$.

The basis is formed by eigenstates such as $\bar{b}b|\lambda\rangle = \lambda|\lambda\rangle$, $\cdots, b^n|\lambda\rangle, \cdots, b|\lambda\rangle, |\lambda\rangle, \bar{b}|\lambda\rangle, \cdots, \bar{b}^n|\lambda\rangle, \cdots$.

Eigenvalues of $\bar{b}b$ are $\cdots, \lambda - n, \cdots, \lambda - 1, \lambda, \lambda + 1, \cdots, \lambda + n, \cdots$.

For $\sigma = +1$, the spectrum stops at the left with $\lambda \geq 0$, and $\mathrm{sign}\langle\lambda+1|\lambda+1\rangle = \mathrm{sign}\langle\lambda|\lambda\rangle$.

For $\sigma = -1$, the spectrum stops at the right with $\lambda \leq -1$, and $\mathrm{sign}\langle\lambda-1|\lambda-1\rangle = -\mathrm{sign}\langle\lambda|\lambda\rangle$.

The eigenstates satisfy $H_\sigma|\lambda + \sigma n\rangle = E_\sigma(n)|\lambda + \sigma n\rangle$, where $E_\sigma(n) = n + \frac{1}{2} + \sigma\lambda$.

The coordinate eigenvalue is $\bar{q}_\sigma = \sigma q_\sigma$.

The identity operator is expressed as $\mathbb{1} = \int dq|\sigma q\rangle\langle q| = \int dp|\sigma p\rangle\langle p|$.

The total Hamiltonian is $H = H_+ + H_-$, with $H_\sigma = \sigma\left(\frac{p_\sigma^2}{2m_\sigma} + \frac{m_\sigma \omega_\sigma^2}{2} q_\sigma^2\right)$.

The classical Hamiltonian is given by

$$H(q,q',p,p') = \frac{\langle q,-q'|H|p,p'\rangle}{\langle q,-q'|p,p'\rangle} = \frac{\langle q|H_+|p\rangle}{\langle q|p\rangle} + \frac{\langle -q'|H_-|p'\rangle}{\langle -q'|p'\rangle}$$

$$= \frac{p_+^2}{2m_+} + \frac{m_+\omega_+^2}{2}q_+^2 - \frac{p_-^2}{2m_-} - \frac{m_-\omega_-^2}{2}q_-^2.$$

The Lagrangian is $L(q,q',p,p') = \frac{\dot{q}_+^2}{2m_+} - \frac{m_+\omega_+^2}{2}q_+^2 - \frac{\dot{q}_-^2}{2m_-} + \frac{m_-\omega_-^2}{2}q_-^2$.
Convergence is achieved by $\omega_\sigma \to \omega_\sigma - \sigma i\epsilon$.

B. Field Operator

The field operator is described by the following equations:

$$M^2 = \ell^{-2} \tag{6}$$

$$L = \frac{1}{2}\partial_\mu\phi_+\partial^\mu\phi_+ - \frac{m^2}{2}\phi_+^2 - \frac{1}{2}\partial_\mu\phi_-\partial^\mu\phi_- - \frac{M^2}{2}\phi_-^2 \tag{7}$$

Here, M^2 represents a characteristic constant associated with the field.
The commutation relations for creation and annihilation operators are given by:

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 2\omega_\mathbf{p}\delta(\mathbf{p}-\mathbf{p}') \tag{8}$$
$$\left[b(\mathbf{p}), b^\dagger(\mathbf{p}')\right] = -(2\pi)^3 2\Omega_\mathbf{p}\delta(\mathbf{p}-\mathbf{p}') \tag{9}$$
$$\left[a(\mathbf{p}), b^\dagger(\mathbf{p}')\right] = [a(\mathbf{p}), b(\mathbf{p}')] = 0 \tag{10}$$

These relations govern the creation and annihilation of particles associated with the field.
The frequencies $\omega_\mathbf{p}$ and $\Omega_\mathbf{p}$ are defined as:

$$\omega_\mathbf{p} = \sqrt{m^2+\mathbf{p}^2}, \quad \Omega_\mathbf{p} = \sqrt{\mathbf{p}^2 - M^2} \tag{11}$$

The vacuum states for the operators are such that $a(\mathbf{p})|0\rangle = b^\dagger(\mathbf{p})|0\rangle = 0$.
The field $\phi(x)$ is decomposed into positive and negative norm subspaces:

$$\begin{aligned}\phi_+(x) &= \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2-m^2)a(k)e^{-ikx} \\ &= \int_{m,\mathbf{k}} [a(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx}] \\ \phi_-(x) &= \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2+M^2)b^\dagger(k)e^{-ikx} \\ &= \int_{M,\mathbf{k}} [b(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{ikx}]\end{aligned} \tag{12}$$

Finally, integrals involving functions f_k are defined over momentum space:

$$\int_{m,\mathbf{k}} f_k = \int \frac{d^3k}{(2\pi)^3 2\omega_\mathbf{k}} f_{\omega_\mathbf{k},\mathbf{k}}, \quad \int_{M,\mathbf{k}} k_k = \int_{\mathbf{k}^2>|M^2|} \frac{d^3k}{(2\pi)^3 2\Omega_\mathbf{k}} f_{\Omega_\mathbf{k},\mathbf{k}} \tag{13}$$

C. Path Integral

The path integral representation for the generating functional is given by:

$$\begin{aligned} e^{iW[\hat{j}]} &= \mathrm{Tr}\, T\left[e^{-i\int_{t_i}^{t_f} dt'[H(t')-j_+(t')\phi_+(t')]}|0\rangle\langle 0|\bar{T}[e^{i\int_{t_i}^{t_f} dt'[H(t')+j_-(t')\phi_-(t')]}]\right] \\ &= \int D[\hat{\phi}]e^{\frac{i}{2}\hat{\phi}\cdot\hat{D}^{-1}\cdot\hat{\phi}+i\hat{j}\cdot\hat{\phi}} \\ &= e^{-\frac{i}{2}\hat{j}\cdot\hat{D}\cdot\hat{j}} \end{aligned} \tag{14}$$

Here, $W[\hat{j}]$ represents the generating functional, $H(t)$ is the Hamiltonian, and $\phi_\pm(t)$ are field components.

The second functional derivative of $W[\hat{j}]$ with respect to the source terms $j_\pm(t)$ gives the connected two-point functions:

$$i\frac{\delta^2 W[\hat{j}]}{\delta i j_a^+ \delta i j_b^+} = iD_{ab}^{++} \tag{15}$$

$$i\frac{\delta^2 W[\hat{j}]}{\delta i j_a^- \delta i j_b^-} = -iD_{ba}^{++*} \tag{16}$$

$$i\frac{\delta^2 W[\hat{j}]}{\delta i j_a^- \delta i j_b^+} = iD_{ab}^{-+} \tag{17}$$

$$i\frac{\delta^2 W[\hat{j}]}{\delta i j_a^+ \delta i j_b^-} = -iD_{ab}^{-+*} \tag{18}$$

The commutation relations are expressed in terms of the correlation functions, and D^{++} and D^{-+} are components of the Green's function matrix.

The trace of time-ordered and anti-time-ordered products, as well as their difference, are defined as:

$$T[\phi_a\phi_b] + \bar{T}[\phi_a\phi_b] = \phi_a\phi_b + \phi_b\phi_a \tag{19}$$

$$D - D^\dagger = D^{+-} - D^{+-*} \tag{20}$$

The components of the matrix $i\begin{pmatrix} D & D^{+-} \\ D^{-+} & D^{--}\end{pmatrix}$ are expressed in terms of correlation functions and imaginary parts:

$$i\begin{pmatrix} D & D^{+-} \\ D^{-+} & D^{--}\end{pmatrix}_{x,y}^{j,k} = i\begin{pmatrix} D^n + i\Im D & -D^f + i\Im D \\ D^f + i\Im D & -D^n + i\Im D\end{pmatrix}_{x,y}^{j,k} \tag{21}$$

The correlation functions are given by:

$$\begin{aligned} iD_{x,x'} &= \Theta(t-t')\langle 0|\phi_x\phi_{x'}|0\rangle + \Theta(t'-t)\langle 0|\phi_{x'}\phi_x|0\rangle \\ 2\Re D_{x,x'} &= 2D^n_{x,x'} = -\epsilon(t-t')i\langle 0|[\phi_x,\phi_{x'}]|0\rangle \\ 2\Im D_{x,x'} &= -\langle 0|\{\phi_x,\phi_{x'}\}|0\rangle \\ iD^{-+}_{x,x'} &= \langle 0|\phi_x\phi_{x'}|0\rangle \\ 2\Re D^{-+}_{x,x'} &= 2D^f_{x,x'} = -i\langle 0|\phi_x\phi_{x'}|0\rangle \\ 2\Im D^{-+}_{x,x'} &= -\langle 0|\{\phi_x,\phi_{x'}\}|0\rangle \\ iD^{\overset{r}{a}}_{x,x'} &= \pm\Theta(\pm(t-t'))\langle 0|[\phi_x,\phi_{x'}]|0\rangle \\ D^{\overset{r}{a}}(x) &= \pm\Theta(\pm t)[D^{-+}(x) - D^{-+}(-x)] \end{aligned} \tag{22}$$

D. Explicit Expressions

The explicit expressions for the Green's functions are given by:

$$
\begin{aligned}
iD^{-+}(x,x') &= \frac{\langle 0|\phi(x)\phi(x')|0\rangle}{\langle 0|0\rangle} \\
&= \int_{m,\mathbf{k},\mathbf{k}'} \frac{\langle 0|[a(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}][a(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}'} + a^\dagger(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}'}]|0\rangle}{\langle 0|0\rangle} \\
&\quad + \int_{M,\mathbf{k},\mathbf{k}'} \frac{\langle 0|[b(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} + b^\dagger(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}][b(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}'} + b^\dagger(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}'}]|0\rangle}{\langle 0|0\rangle} \\
&= \int_{m,\mathbf{k}} e^{-i\omega_{\mathbf{k}}(t-t')+i\mathbf{k}(\mathbf{x}-\mathbf{x}')} + \int_{M,\mathbf{k}} e^{i\Omega_{\mathbf{k}}(t-t')-i\mathbf{k}(\mathbf{x}-\mathbf{x}')}
\end{aligned} \tag{23}
$$

The corresponding function $iD^{+-}(x,x')$ is simply the complex conjugate of $iD^{-+}(x',x)$.
The causal and anti-causal parts of the Green's functions are given by:

$$
\begin{aligned}
\Theta(t-t')D^{-+}(x,x') &= \int_\omega \frac{e^{-i\omega(t-t')}}{\omega+i\epsilon} \int_{m,\mathbf{k}} e^{-i\omega_{\mathbf{k}}(t-t')+i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \\
&\quad + \int_\omega \frac{e^{i\omega(t-t')}}{-\omega+i\epsilon} \int_{M,\mathbf{k}} e^{i\Omega_{\mathbf{k}}(t-t')-i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \\
&= \int_{m,\mathbf{k},\omega} \frac{e^{-i(\omega_{\mathbf{k}}+\omega)(t-t')+i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{\omega+i\epsilon} \\
&\quad - \int_{M,\mathbf{k},\omega} \frac{e^{i(\omega+\Omega_{\mathbf{k}})(t-t')-i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{\omega-i\epsilon}
\end{aligned} \tag{24}
$$

Similarly, for the anti-causal part:

$$
\begin{aligned}
\Theta(t'-t)D^{+-}(x,x') &= -\int_{m,\mathbf{k},\omega} \frac{e^{-i\omega(t-t')+i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{\omega+\omega_{\mathbf{k}}-i\epsilon} \\
&\quad + \int_{M,\mathbf{k},\omega} \frac{e^{i\omega(t-t')-i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{\omega+\Omega_{\mathbf{k}}+i\epsilon}
\end{aligned} \tag{25}
$$

The complete Green's function $D(x,x')$ is obtained by combining these causal and anti-causal parts:

$$
\begin{aligned}
D(x,x') &= \Theta(t-t')D^{-+}(x,x') + \Theta(t'-t)D^{+-}(x,x') \\
&= \int_k \left[\frac{e^{-ik\cdot x}}{k^2-m^2+i\epsilon} - \frac{\Theta(\mathbf{k}^2-M^2)e^{-ik\cdot x}}{k^2+M^2-i\epsilon}\right]
\end{aligned} \tag{26}
$$

Finally, the real and anti-chronological parts $D^{\overset{r}{a}}(x)$ are given by:

$$
\begin{aligned}
D^{\overset{r}{a}}(x) &= \pm\Theta(\pm t)\int_k e^{-ik\cdot x}\left[\frac{1}{(k^0\pm i\epsilon)^2-\mathbf{k}^2-m^2} - \frac{\Theta(\mathbf{k}^2-M^2)}{(k^0\pm i\epsilon)^2-\mathbf{k}^2+M^2}\right] \\
&= \pm\Theta(\pm t)\int_k e^{-ik\cdot x}\left[\frac{1}{(k^0\pm i\epsilon-\omega_{\mathbf{k}})(k^0\pm i\epsilon+\omega_{\mathbf{k}})} - \frac{\Theta(\mathbf{k}^2-M^2)}{(k^0\pm i\epsilon-\Omega_{\mathbf{k}})(k^0\pm i\epsilon+\Omega_{\mathbf{k}})}\right] \\
&= \pm\Theta(t)i\int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\left[\frac{e^{i\omega_{\mathbf{k}}t}-e^{-i\omega_{\mathbf{k}}t}}{2\omega_{\mathbf{k}}} - \Theta(\mathbf{k}^2-M^2)\frac{e^{i\omega_{\mathbf{k}}t}-e^{-i\omega_{\mathbf{k}}t}}{2\Omega_{\mathbf{k}}}\right]
\end{aligned} \tag{27}
$$

These expressions provide a detailed insight into the structure and behavior of the Green's functions in the given quantum field theory.

E. Real Space Expressions for $m = 0$

For the case of $m = 0$, where the mass term is absent, the contribution at large k is negligible in a cutoff theory. Let's examine the expressions for the Green's functions in this scenario:

The propagator $D(x)$ is given by:

$$
\begin{aligned}
D(x) = D_{x0} &\to \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}-\epsilon' k} \int_{\omega} \frac{e^{-i\omega t}}{\omega^2 - k^2 + i\epsilon} \\
&= -\frac{i}{4\pi^2} \int dk\, k^2\, dc\, e^{ik(rc+i\epsilon')} \left(\frac{\Theta(t) e^{-i(k-i\epsilon)t}}{2k - i\epsilon} + \frac{\Theta(-t) e^{i(k-i\epsilon)t}}{2k - i\epsilon} \right) \\
&= -\frac{i}{8\pi^2 r} \int_0^\infty dk\, k\, \frac{e^{ik(r-i\epsilon')}}{k - i\epsilon} e^{-i(k-i\epsilon)|t|} \\
&= -\frac{1}{4\pi^2 (r^2 - (|t| - i\epsilon')^2)} \\
&= \frac{i}{4\pi^2 x^2 - i\epsilon'} \\
&= P \frac{i}{4\pi^2 x^2} - \frac{1}{4\pi} \delta(x^2),
\end{aligned} \tag{28}
$$

where P denotes the principal value of the integral.

Similarly, for $D^{\overset{r}{a}}(x)$:

$$
\begin{aligned}
D^{\overset{r}{a}}(x) = D^{\overset{r}{a}}_{x0} & \\
&= -\frac{\Theta(t)}{2(2\pi)^2 r} \int dk k^2 \frac{e^{ikr} - e^{-ikr}}{ikr} \frac{e^{-ikt} - e^{ikt}}{2k} \\
&\quad - \frac{\Theta(-t)}{2(2\pi)^2 r} \int dk k^2 \frac{e^{ikr} - e^{-ikr}}{ikr} \frac{e^{-ikt} - e^{ikt}}{2k} e^{-\epsilon|t|} \\
&= -\frac{1}{8\pi r} \int_{-\infty}^{\infty} \frac{dk}{2\pi} (e^{ik(r-t)} + e^{ik(-r+t)} - e^{ik(-r-t)} - e^{ik(r+t)})(\Theta(t) - \Theta(-t)) e^{-\epsilon|t|} \\
&= -\frac{\Theta(t)\delta(r-t) + \Theta(-t)\delta(r+t)}{4\pi r} e^{-\epsilon|t|} \\
&= -\frac{\Theta(\pm t)\delta(t \mp r)}{4\pi r} e^{-\epsilon|t|} \\
&= -\Theta(\pm t) \frac{\delta(t^2 - r^2)}{2\pi} e^{-\epsilon|t|}.
\end{aligned} \tag{29}
$$

Finally, the regular and non-regular parts are:

$$
D^f(x) = \frac{1}{2}(D^{\overset{r}{a}}(x) - D^{\overset{r}{a}}(x)) = -\frac{1}{4\pi} \delta(x^2) \epsilon(x^0), \tag{30}
$$

$$
D^n(x) = \frac{1}{2}(D^{\overset{r}{a}}(x) + D^{\overset{r}{a}}(x)) = -\frac{1}{4\pi} \delta(x^2). \tag{31}
$$

These expressions unveil the structure of the Green's functions in a massless theory, particularly emphasizing the behavior of the propagator in real space.

F. Real Space Expressions for $m \neq 0$

For a non-zero mass term ($m \neq 0$), the propagator $D(k^2)$ is given by:

$$
D(k^2) = -i \, \mathrm{sign}(\epsilon) \int_0^\infty ds e^{i \, \mathrm{sign}(\epsilon) s (k^2 - m^2 + i\epsilon)}. \tag{32}
$$

Utilizing the result that is analytic for $\operatorname{Im} e^{i\theta} > 0$:

$$\int dx e^{\frac{i}{2} a e^{i\theta} x^2} = \sqrt{\frac{2\pi}{a}} e^{i\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}, \tag{33}$$

and for real a:

$$\int \frac{d^4k}{(2\pi)^4} e^{(ia-\epsilon)k^2 - ikx} = -\frac{i\,\mathrm{sign}(a)}{32\pi^2 a^2} e^{-\frac{ix^2}{4a}}, \tag{34}$$

the expression for $D(x)$ becomes:

$$\begin{aligned} D(x) &= -i\,\mathrm{sign}(\epsilon) \int \frac{d^4k}{(2\pi)^4} \int_0^\infty ds e^{i\,\mathrm{sign}(\epsilon')s(k^2 - m^2 + i\epsilon) - ikx} \\ &= -\frac{1}{32\pi^2} \int \frac{da}{a^2} e^{-i\,\mathrm{sign}(\epsilon)a(m^2 - i\epsilon) - \frac{ix^2}{4a}} \\ &= \frac{1}{8\pi^2} \int_{-\infty}^\infty d\alpha e^{-\frac{im^2 - i\epsilon}{4\alpha} - ix^2\alpha} \\ &= \frac{1}{4\pi^2} \int_0^\infty d\alpha \cos\left(\frac{m^2}{4\alpha} + x^2\alpha\right) \\ &= \frac{1}{4\pi^2} \partial_{x^2} \int_0^\infty \frac{d\alpha}{\alpha} \sin\left(\frac{m^2}{4\alpha} + x^2\alpha\right) \\ &= \frac{\Theta(x^2)}{4\pi^2} \partial_{x^2} \int_{-\infty}^\infty d\theta \sin(\sqrt{(m^2 - i\epsilon)x^2} \cosh\theta) \\ &= \frac{\Theta(x^2)}{4\pi} \partial_{x^2} J_0(\sqrt{(m^2 - i\epsilon)x^2}), \end{aligned} \tag{35}$$

where J_0 is the Bessel function of the first kind with order 0.
Now, considering $D^{\overset{r}{a}}(x)$ for positive mass square:

$$D^{\overset{r}{a}}(x) = -\frac{\Theta(t)}{2(2\pi)^2 r} \left[\int_m^\infty \frac{dkk}{\omega_k} (e^{ikr} - e^{-ikr})(e^{-i\omega_k t} - e^{i\omega_k t}) - i \int_0^m \frac{dkk}{\omega_k} (e^{ikr} - e^{-ikr}) e^{-\omega_k t} \right]. \tag{36}$$

For negative mass square, where $\tilde{\omega}_{\boldsymbol{k}} = +\sqrt{|m^2 - \boldsymbol{k}^2|} > 0$:

$$D^r(x) = -\frac{\Theta(t)}{2(2\pi)^2 r} \left[\int_m^\infty \frac{dkk}{\tilde{\omega}_k} (e^{ikr} - e^{-ikr})(e^{-i\tilde{\omega}_k t} - e^{i\tilde{\omega}_k t}) - i \int_0^m \frac{dkk}{\tilde{\omega}_k} (e^{ikr} - e^{-ikr}) e^{-\tilde{\omega}_k t} \right]. \tag{37}$$

These expressions provide insights into the behavior of the propagators for non-zero mass in real space.

IV. CONCLUSION

In conclusion, the exploration of Quasi Photons within the framework of an unconventional quantum field theory has unfolded intricate mathematical facets, reshaping our conceptual understanding of the quantum realm. The model, characterized by a harmonic oscillator with negative norm and a massless scalar field with negative mass squared in the subspace with negative norm, challenges conventional notions of photons, ushering in a profound transformation.
The real-space expressions for the propagator, particularly in the limit of zero mass ($m = 0$), reveal captivating mathematical structures. The resulting expression for the propagator $D(x)$ embodies both principal value and delta function terms, exemplifying the rich interplay between mathematical sophistication and physical insight:

$$D(x) = P\frac{i}{4\pi^2 x^2} - \frac{1}{4\pi}\delta(x^2).$$

Furthermore, the real-space representation of the advanced and retarded propagators, denoted as $D^r(x)$ and $D^a(x)$ respectively, showcases intricate mathematical features:

$$D^{\overset{r}{a}}(x) = -\frac{i}{4\pi^2(x^2 - i\epsilon')} + \frac{1}{4\pi}\delta(x^2) - \Theta(t)R + \Theta(-t)A,$$

where R and A encapsulate the intricacies of the advanced and retarded contributions, each exhibiting a delicate balance of mathematical elegance and physical significance. In summary, the mathematical tapestry woven in the exploration of Quasi Photons not only enriches our understanding of unconventional quantum field theories but also beckons further inquiry into the profound interplay between mathematical formalism and the nature of quantum phenomena. This endeavor prompts a reassessment of our conceptual framework, urging us to perceive the quantum world through a lens that embraces both mathematical sophistication and physical insight.

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