Separation Theorems in Hilbert Spaces as Bases of Convex Programming

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SEPARATION THEOREMS IN HILBERT SPACES AS BASES OF CONVEX PROGRAMMING

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Abstract. Separation theorems are the bases of convex programming. They are important consequences of Hahn-Banach theorem. This work begins considering vector spaces in broad sense. Then normed spaces and finally Hilbert spaces. It ends with the consideration of these results in convex programming, an important quantitative tool in operations research, finance, management and economics.

Keywords: Hahn-Banach theorem, separation theorems, convex programming, Kuhn-Tucker theorem.

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1 Introduction

The Hahn-Banach theorem is presented, with great generality, together with an important separation theorem. Then these results are particularized: first for normed spaces and then for a subclass of these spaces: the Hilbert spaces.
The richness of these results is emphasized in the last section where it is shown that they permit to obtain very important results in the applications: the Kuhn-Tucker theorem, in finite and infinite dimensions, the convex programming main results, so important in operations research, finance, management and economics. More works around this subject are, for example, $[4-16]$.

2 The Hahn-Banach Theorem

Definition 2.1
Consider a vector space L and its subspace L_0. Suppose that in L_0 it is defined a linear functional f_0. A linear functional f defined in the whole space L is an extension of the functional f_0 if and only if $f(x) = f_0(x), \forall_{x \in L_0}$. ∎

Theorem 2.1 (Hahn-Banach**)**

Be p a positively homogeneous convex functional, defined in a real vector space L, and L_0 an L subspace. If f_0 is a linear functional defined in L_0, fulfilling the condition

$$f_0(x) \leq p(x), \underset{x \in L_0}{\forall} \quad (2.1),$$

there is an extension f of f_0 defined in L, linear, and such that $f(x) \leq p(x), \underset{x \in L}{\forall}$.

Demonstration:

Begin demonstrating that if $L_0 \neq L$, there is an extension of f_0, f' defined in a subspace L', such that $L \subset L'$, in order to fulfill the condition (2.1).

Be $z \in L - L_0$; if L' is the subspace generated by L_0 and z, each point of L' is expressed in the form $tz + x$, being $x \in L_0$. If f' is an extension (linear) of the functional f_0 to L', it will happen that $f'(tz + x) = tf'(z) + f_0(x)$ or, making $f'(z) = c$, $f'(tz + x) = tc + f_0(x)$. Now choose c, fulfilling the condition (2.1) in L', that is: in order that the inequality $f_0(x) + tc \leq p(x + tz)$, for any $x \in L_0$ and any real number t, is accomplished. For $t > 0$ this inequality is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \leq p\left(\frac{x}{t} + z\right)$ or

$$c \leq p\left(\frac{x}{t} + z\right) - f_0\left(\frac{x}{t}\right) \qquad (2.2).$$

For $t < 0$ it is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \geq -p\left(-\frac{x}{t} - z\right)$, or

$$c \geq -p\left(-\frac{x}{t} - z\right) - f_0\left(\frac{x}{t}\right) \qquad (2.3).$$

Now it will be demonstrated that there is always c satisfying simultaneously the conditions (2.2) and (2.3).

Considering any y' and y'' belonging to L_0,

$$-f_0(y'') + p(y'' + z) \geq -f_0(y') - p(-y' - z) \qquad (2.4),$$

as $f_0(y'') - f_0(y') \leq p(y'' - y') = p\big((y'' + z) - (y' + z)\big) \leq p(y'' + z) + p(-y' - z)$. Be $c'' = \inf\limits_{y''}\big(-f_0(y'') + p(y'' + z)\big)$ and $c' = \sup\limits_{y'}\big(-f_0(y') - p(-y' - z)\big)$. As y' and y'' are arbitrary, it results from (2.4) that $c'' \geq c'$. Choosing c in order that $c'' \geq c \geq c'$, it is defined the functional f' on L' through the formula $f'(tz + x) = tc + f_0(x)$. This functional satisfies the condition (2.1). So any functional f_0 defined in a subspace $L_0 \subset L$ and subject in L_0 to the condition (2.1), may be extended to a subspace L'. The extension f' satisfies the condition $f'(x) \leq p(x)$,

$\underset{x \in L'}{\forall}$. If L has an algebraic numerable base $(x_1, x_2, \ldots, x_n, \ldots)$ the functional in L is built by finite induction, considering the increasing sequence of subspaces $L^{(1)} = (L_0, x_1), L^{(2)} = \left(L^{(1)}, x_2\right), \ldots$ designating $\left(L^{(k)}, x_{k+1}\right)$ the L subspace generated by $L^{(k)}$ and x_{k+1}. In the general case, that is, when L has not an algebraic numerable base, it is mandatory to use a transfinite induction process, for instance the Haudsdorf maximal chain theorem.

So call $\mathcal{F}$ the set of the whole pairs(L', f'), at which L' is an L subspace that contains L_0 and f' is an extension of f_0 to L' that fulfills (2.1). Order partially $\mathcal{F}$so that $(L', f') \leq (L'', f'')$ if and only if $L' \subset L''$and $f''_{|L'} = f'$. By the Haudsdorf maximal chain theorem, there is a chain, that is: a subset of $\mathcal{F}$ totally ordered, maximal, that is: not strictly contained in another chain. Call it Ω. Be Φ the family of the whole L' such that $(L', f') \in \Omega$. Φ is totally ordered by the sets inclusion; so, the union T of the whole elements of Φ is a L subspace. If $x \in \mathrm{T}$ then $x \in L'$ for some $L' \in \Phi$; define $\tilde{f}(x) = f'(x)$, where $f'$is the extension of f_0 that is in the pair (L', f')- the definition of $\tilde{f}$ is obviously coherent. It is easy to check that $\mathrm{T} = L$ and that $f = f'$ satisfies the condition (2.1).■

Then it will be presented the Hahn-Banach theorem complex case, the Hahn contribution to the theorem.

Theorem 2.2 (Hahn-Banach**)**

Be p an homogeneous convex functional defined in a vector space L and f_0 a linear functional, defined in a subspace $L_0 \subset L$, satisfying the condition$|f_0(x)| \leq p(x), x \in L_0$. Then, there is a linear functional *f* defined in L, satisfying the conditions

$|f(x)| \leq p(x), x \in L; f(x) = f_0(x), x \in L_0$.

Demonstration:

Call L_R and L_{0R} the real vector spaces underlying, respectively, the spaces L and L_0 . Clearly, p is an homogeneous convex functional in L_R and $f_{0R}(x) = Ref_0(x)$ a real linear functional in L_{0R} fulfilling the condition $|f_{0R}(x)| \leq p(x)$ and so, $f_{0R}(x) \leq p(x)$. Then, owing to Theorem 2.1, there is a real linear functional f_R, defined in the whole L_R space, that satisfies the conditions $f_R(x) \leq p(x), x \in L_R$; $f_R(x) = f_{0R}(x), x \in L_{0R}$. But, $-f_R(x) = f_R(-x) \leq p(-x) = p(x)$, and

$$|f_R(x)| \leq p(x), x \in L_R \qquad (2.5).$$

Define in L the functional f making $f(x) = f_R(x) - if_R(ix)$. It is immediate to conclude that f is a complex linear functional in L such that $f(x) = f_0(x), x \in L_0; Ref(x) = f_R(x), x \in L$.

It is only missing to demonstrate that $|f(x)| \leq p(x), \underset{x \in L}{\forall}$.

Proceed by absurd: suppose that there is $x_0 \in L$ such that $|f(x_0)| > p(x_0)$. So, $f(x_0) = \rho e^{i\varphi}, \rho > 0$, and making $y_0 = e^{-i\varphi}x_0$, it would happen that $f_R\,(y_0) = Re\left[e^{-i\varphi}f(x_0\,)\right] = \rho > p(x_0) = p(y_0)$ that is conflicting to (2.5).■

3 Vector Spaces Convex Parts Separation

The next theorem, a very useful consequence of the Hahn-Banach theorem, is about vector space convex parts separation. Beginning with

Definition 3.1

Be M and N two subsets of a real vector space L. A linear functional f defined in L separates M and N if and only if there is a number c such that $f(x) \geq c$, for $x \in M$ and $f(x) \leq c$, for $x \in N$ that is, if $\inf_{x\in M} f(x) \geq \sup_{x\in N} f(x)$. A functional f separates strictly the sets M and N if and only if $\inf_{x\in M} f(x) > \sup_{x\in N} f(x)$. ■

Theorem 3.1 (Separation**)**

Suppose that *M* and *N* are two convex subsets of a vector space *L* such that the kernel of at least one of them, for instance *M*, is non-empty and does not intersect the other set; so, there is a linear functional non-null on *L* that separates *M* and *N*.

Demonstration:

Less than on translation, it is supposable that the point 0 belongs to the kernel of *M*, which is designated $\dot{M}$. So, given $y_0 \in N$, $-y_0$ belongs to the kernel of $M - N$ and 0 to the kernel of $M - N + y_0$. As $\dot{M} \cap N = \emptyset$, by hypothesis, 0 does not belong to the kernel of $M - N$ and y_0 does not belong to the one of $M - N + y_0$. Put $K = M - N + y_0$ and be p the Minkovsky functional of $\dot{K}$. So $p(y_0) \geq 1$, since $y_0 \notin \dot{K}$. Define, now, the linear functional $f_0(\alpha y_0) = \alpha p(y_0)$. Note that f_0 is defined in a space with dimension1, constituted by elements αy_0, and it is such that $f_0(\alpha y_0) \leq p(\alpha y_0)$. In fact, $p(\alpha y_0) = \alpha p(y_0)$, when $\alpha \geq 0$ and $f_0(\alpha y_0) = \alpha f_0(y_0) < 0 < p(\alpha y_0)$, when $\alpha > 0$. Under these conditions, after the Hahn-Banach theorem, it is possible to state the existence of linear functional f , defined in L, that extends f_0, and such that $f(y) \leq p(y), \underset{y\,\in\,L}{\forall}$. Then it results $f(y) \leq 1, \underset{y\,\in\,K}{\forall}$ and $f(y_0) \geq 1$. In consequence:

-f separates the sets K and $\{y_0\}$, that is

- f separates the sets *M*-*N* and $\{y_0\}$, that is

-f separates the sets *M* and *N*.■

4 The Hahn-Banach Theorem for Normed Spaces

Definition 4.1

Consider a continuous linear functional f in a normed space E. It is called f norm, and designated $\|f\|$: $\|f\| = \sup_{\|x\|\leq 1} |f(x)|$ that is: the supreme of the values assumed by $|f(x)|$ in the E unitary ball.■

Observation:

-The continuous linear functionals class, with the norm above defined, is a normed vector space, called the E dual space, designated E'.

The Theorem 2.1 in normed spaces is:

Theorem 4.1 (Hahn-Banach**)**

Call L a subspace of a real normed space E. And f_0 a bounded linear functional in L. So, there is a linear functional defined in E, extension of f_0, such that $\|f_0\|_{L'} = \|f\|_{E'}$.

Demonstration:

It is enough to think in the functional K satisfying $K\|x\| = \|f_0\|_{L'}$. As it is convex and positively homogeneous, it is possible to put $p(x) = K\|x\|$ and to apply Theorem 2.1.■

Observation:

-To see an interesting geometric interpretation of this theorem, consider the equation $\|f_0(x)\| = 1$. It defines, in L, an hiperplane at distance $\frac{1}{\|f_0\|}$ of 0. Considering the $f_0$extension f, with norm conservation, it is obtained an hiperplane in E, that contains the hiperplane considered behind in L, at the same distance from the origin.

The Theorem 2.2 in normed spaces is:

Theorem 4.2 (Hahn-Banach**)**

Be E a complex normed space and f_0 a bounded linear functional defined in a subspace $L \subset E$. So, there is a bounded linear functional f, defined in E, such that $f(x) = f_0(x), x \in L; \|f\|_{E'} = \|f_0\|_{L'}$. ■

Two separation theorems, important consequences of the Hahn-Banach theorem, applied to the normed vector spaces, are so presented:

Theorem 4.3 (Separation**)**

Consider two convex sets A and B in a normed space E. If one of them, for instance A, has at least on interior point and $(intA) \cap B = \emptyset$, there is a continuous linear functional non-null that separates the sets A and B.■

Theorem 4.4 (Separation**)**

Consider a closed convex set *A*, in a normed space *E*, and a point $x_0 \in E$, not belonging to *A*. So, there is a continuous linear functional, non-null, that separates strictly $\{x_0\}$ and *A*.■

5 Separation Theorems in Hilbert Spaces

The separation theorems, seen in the former section, are effective in Hilbert spaces. But, due to the Riesz representation theorem, surely the most famous representation theorem:

Theorem 5.1 (Riesz representation**)**

In a Hilbert space H, every continuous linear functional $f(\cdot)$ may be represented in the form $f(x) = [x, \tilde{q}]$ where $\tilde{q} = \frac{\overline{f(q)}}{[q,q]} q$. ■

they may be formulated in the following way (from now on, only real Hilbert spaces will be considered):

Theorem 5.2 (Separation**)**

Consider two convex sets *A* and *B* in a Hilbert space *H*. If one of them, for instance *A*, has at least one interior point and $(intA) \cap B = \emptyset$, there is a non-null vector *v* such that $\sup_{x \in A}[v, x] \leq \inf_{y \in B}[v, y]$. ■

Theorem 5.3 (Separation**)**

Consider a closed convex set *A*, in a Hilbert space *H*, and a point $x_0 \in H$, not belonging to *A*. So, there is a non-null vector *v*, such that $[v, x_0] < \inf_{x \in A}[v, x]$. ■

Another separation theorem:

Theorem 5.4 (Separation**)**

Two closed convex subsets *A* and *B*, in a Hilbert space, at finite distance, that is: such that: $\inf_{x \in A, y \in B} \|x - y\| = d > 0$ may be strictly separated: $\inf_{x \in A}[v, x] > \sup_{y \in B}[v, y]$. ■

It is also possible to establish that:

Theorem 5.5 (Separation**)**

Being H a finite dimension Hilbert space, if *A* and *B* are disjoint and non-empty convex sets they always may be separated.■

6 Convex Programming

A class of convex programming problems, at which it is intended to minimize convex functionals subject to convex inequalities, is outlined now. Begin presenting a basic result that characterizes the minimum point of a convex functional subject to convex inequalities. Note that it is not necessary to impose any continuity conditions. Only geometric conditions are important.

Theorem 6.1 (Kuhn-Tucker**)**

Be *f(x)*, $f_i(x), i = 1, \dots, n,$ convex functionals defined in a convex subset *C* of a Hilbert space. Consider the problem $\min_{x \in C} f(x)$, $sub.: f_i(x) \leq 0, i = 1, \dots$ Be x_0 a point where the minimum, supposed finite, is reached. Suppose also that for each vector u in E_n, Euclidean space with dimension *n*, non-null and such that $u_k \geq 0$, there is a point *x* in *C* such that $\sum_1 u_k f_k(x) < 0$,designating u_k the components of *u*. So,

i) There is a vector *v*, with non-negative components $\{v_k\}$, such that

$$\min_{x \in C} \left\{ f(x) + \sum_1^n v_k f_k(x) \right\} = f(x_0) + \sum_1^n v_k f_k(x_0) = f(x_0) \qquad (6.1),$$

ii) For every vector u in E_n with non-negative components, that is: belonging to the positive cone of E_n ,

$$f(x) + \sum_1^n v_k f_k(x)$$

$$\geq f(x_0) + \sum_1^n v_k f_k(x_0) \geq f(x_0) + \sum_1^n u_k f_k(x_0) \quad (6.2). \blacksquare$$

Corollary 6.1 (Lagrange duality**)**

In the conditions of Theorem 6.1 $f(x_0) = \sup_{u \geq 0} \inf_{x \in C} f(x) + \sum_1^n u_k f_k(x). \blacksquare$

Observation:

-This corollary is useful supplying a process to determine the problem optimal solution,

-If the whole v_k in expression (6.2) are positive, x_0 is a point that belongs to the border of the convex set defined by the inequalities,

-If the whole v_k are zero, the inequalities do not influence the problem, that is: the minimum is equal to the one of the restrictions free problem.

Considering non-finite inequalities:

Theorem 6.2 (Kuhn-Tucker in infinite dimension**)**

Be *C* a convex subset of a Hilbert space *H* and *f* (*x*) a real convex functional defined in *C*. Be *I* a Hilbert space with a closed convex cone $\wp$, with non-empty interior, and *F*(*x*) a convex transformation from *H* to *I* (convex in relation to the order introduced by cone

$\wp$: if $x, y \in \wp, x \geq y \; if \; x - y \in \wp$). Be x_0 a f (x) minimizing in C subjected to the inequality $F(x) \leq 0$.Consider $\wp^* = \left\{x: [x, p] \geq 0, \underset{x \in \wp}{\forall}\right\}$ (dual cone). Admit that given any $u \in \wp^*$ it is possible to determine x in C such that $[u, F(x)] < 0$. So, there is an element v in the dual cone $\wp^*$, such that for x in C $f(x) + [v, F(x)] \geq f(x_0) + [v, F(x_0)] \geq f(x_0) + [u, F(x_0)]$, being u any element of $\wp^*$. ∎

Corollary 6.2 (Lagrange duality in infinite dimension**)**

$f(x_0) = \sup_{v \in \wp^*} \inf_{x \in C} (f(x) + [v, F(x)])$ in the conditions of Theorem 6.2. ∎

7 Conclusions

The Hahn-Banach theorem was presented with great generality, real and complex version, followed consequently by an important separation theorem.

These results were specified for normed spaces and then for a subclass of these spaces: the Hilbert spaces. That is: they were rephrased for Hilbert spaces using the Riesz representation theorem.

Examples of the presented results fruitfulness are patent in the former section, where it is shown that they permit to obtain important results, for the applications, as the Kuhn-Tucker theorem in finite and infinite dimension. Now the basic mathematical structures considered were real Hilbert spaces. The problems studied were convex optimization problems in which, as it is well known, the separation theorems are a key tool.

The Kuhn-Tucker theorem is the convex programming main result so important in operations research, finance, management and economics. For instance, it is remarkable, among others, its application in portfolios optimization through quadratic programming. Really, due the particular form assumed by the Kuhn-Tucker conditions, the quadratic programming problems may be solved with the simplex algorithm. This is the so called Frank and Wolfe method, see [5].

References

[1] Aubin, J. P.: *Applied Functional Analysis*. John Wiley & Sons Inc., New York, 1979.

[2] Balakrishnan, A. V.: *Applied Functional Analysis*. Springer-Verlag New York Inc., New York, 1981.

[3] Brézis, H. : *Analyse Fonctionelle (Théorie et Applications)*. Masson, Paris, 1983.

[4] Ferreira, M. A. M.: *Aplicação dos Teoremas de Separação na Programação Convexa em Espaços de Hilbert*. Revista de Gestão, I (2), pp 41-44, 1986.

[5] Ferreira, M. A. M. and Amaral, I.:*Programação Matemática*. Edições Sílabo, Lisboa, 1995.

[6] Ferreira, M. A. M., Andrade, M. and Matos, M. C.: *Separation Theorems in Hilbert Spaces Convex Programming*. Journal of Mathematics and Technology, 1 (5), pp 20-27, 2010.
[7] Ferreira, M. A. M. and Andrade, M.: *Management Optimization Problems*. International Journal of Academic Research, Vol. 3 (2, Part III), pp 647-654, 2011.
[8] Ferreira, M. A. M. and Andrade, M.: *Hahn-Banach Theorem for Normed Spaces*. International Journal of Academic Research, 3 (4, Part I), pp 13-16, 2011a.
[9] Ferreira, M. A. M. and Andrade, M.: *Riesz Representation Theorem in Hilbert Spaces Separation Theorems*. International Journal of Academic Research, 3 (6, II Part), pp 302-304, 2011b.
[10] Ferreira, M. A. M. and Andrade, M.: *Separation of a Vector Space Convex Parts*. International Journal of Academic Research, 4 (2), pp 5-8, 2012.
[11] Ferreira, M. A. M. and Filipe, J. A.: *Convex Sets Strict Separation in Hilbert Spaces*. Applied Mathematical Sciences, 8 (61-64), 3155-3160, 2014. http://dx.doi.org/10.12988/ams.2014.44257
[12] Ferreira, M. A. M. and Matos, M. C. P.:*Convex Sets Strict Separation in the Minimax Theorem*. Applied Mathematical Sciences, 8 (33-36), 1781-1787, 2014. http://dx.doi.org/10.12988/ams.2014.4271
[13] Ferreira, M. A. M., Andrade, M. and Filipe, J. A.: *Kuhn-Tucker's Theorem for inequalities in Infinite Dimension*. Journal of Mathematics and Technology, 3 (1), pp 57-60, 2012.
[14] Ferreira, M. A. M., Andrade, M. and Filipe, J. A.: *Weak Convergence in Hilbert Spaces*. International Journal of Academic Research, 4 (4), pp 34-36, 2012.
[15]Ferreira, M. A. M., Andrade, M. and Filipe, J. A.: *The Concept of Weak Convergence in Hilbert Spaces*.12th Conference on Applied Mathematics, APLIMAT 2013, Proceedings. 12th Conference on Applied Mathematics, APLIMAT 2013; Bratislava; Slovakia; 5 February 2013 through 7 February 2013.
[16] Ferreira, M. A. M., Andrade, M, Matos, M. C., Filipe, J. A. and Coelho, M.: *Minimax Theorem and Nash Equilibrium*. International Journal of Latest Trends in Finance & Economic Sciences, 2(1), pp 36-40, 2012.
[17] Kakutani, S.: *A Generalization of Brouwer's Fixed Point Theorem*. Duke Mathematics Journal, 8, 1941.
[18] Kantorovich, L. V. and Akilov, G. P.: *Functional Analysis*. Pergamon Press, Oxford, 1982.
[19] Kolmogorov, A. N. and Fomin, S. V.: *Elementos da Teoria das Funções e de Análise Funcional*. Editora Mir, 1982.
[20] Matos, M. C. and Ferreira, M. A. M.: *Game Representation -Code Form*. Lecture Notes in Economics and Mathematical Systems; 567, pp 321-334, 2006.
[21] Matos, M. C., Ferreira, M. A. M. and Andrade, M.: *Code Form Game*. International Journal of Academic Research, 2(1), pp 135-141, 2010.
[22] Matos, M. C., Ferreira, M. A. M., Filipe, J. A. and Coelho, M.: *Prisonner`s Dilemma: Cooperation or Treason?*. PJQM-Portuguese Journal of Quantitative Methods, Vol. 1(1), pp 43-52, 2010.
[23] Nash, J.: *Non-Cooperative Games*. Annals of Mathematics, 54, 1951.
[24] Neumann, J. von and Morgenstern, O.: *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, New Jersey, 1947.
[25] Neumann, J. von and Morgenstern, O.: *Theory of Games and Economic Behavior*. John Wiley & Sons Inc., New York, 1967.
[26] Royden, H. L.: *Real Analysis*. Mac Milan Publishing Co. Inc, New York, 1968.