# Canonical Ternary Quadratic Forms: Linear Algebraic Approach

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# “Canonical” Ternary Quadratic Forms: Linear Algebraic Approach:

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**ABSTRACT**

In this research paper, it is reasoned that “canonical”/interesting ternary quadratic forms ( in the spirit of Ramanujan ternary quadratic form) fall in 2 distinct quadratic form classes. Two interesting theorems are proved which show that when the eigenvalues of symmetric matrices associated with the canonical ternary forms are integers, the ternary quadratic form can never equal the square of an integer. Alongwith Fermat’s last number Theorem, these Theorems provide new results on representation of integers using higher degree ternary forms associated with the 2 x 2 symmetric matrix. Generalization of the results to quadratic/higher degree forms associated with higher dimensional square matrix are discussed.

## 1. Introduction:

Based on the concept of “zero”, algebraic symbolism enabled mathemaaticians to propose algebraic equations of utility in real world applications. Specifically, algebraic equations in which the variables are allowed to assume integer values were proposed by Diophantus. In geometric applications, Pythagorus proved that the following Diophantine equation is satisfied by the sides { x, y, z } of a right angle triangle i.e.

$$x^2 + y^2 = z^2.$$

Diophantus studied linear algebraic equations in two variables and explored their solutions using the Greatest Common Divisor (GCD) algorithm [1]. The results were documented in the book “arithmetica” by Diophantus. Fermat acquired a copy of arithmetica and proved several interesting results leading to the field of NUMBER THEORY. In fact, Euclid also proved several interesting theorems in elementary number theory. Fermat proved that every prime ‘p’ of the form $p \equiv 1 (mod)\, 4$ can be uniquely expressed as the sum of squares of two positive integers, where as a prime of the form $p \equiv 3(mod)\, 4$ can never be expressed in such fashion. This theorem ( proved based on the method of infinite descent ) provided the starting point for the theory of binary quadratic forms [2]. In the spirit of the Fermat’s theorem, Lagrange proved that every positive integer can be expressed as the sum of squares of four integers. Also, Jacobi proved the so called “4-square theorem” dealing with expression of a positive integer as sum of 4 squares. Legender initiated the theory of Ternary quadratic forms in three variables of the form

$$x^2 + y^2 + z^2 = m, \quad where\ x, y, z, m\ assume\ integer\ values.$$

Lagrange was the first one to show that a quaternary quadratic form of the type

$$x^2 + y^2 + z^2 + w^2, \ where\ x, y, z, w\ are\ integers$$

is UNIVERSAL ( in the sense that every positive integer can be expressed using such quaternary quadratic form ). Ramanujan identified 54 quaternary quadratic forms of the type

$$a\, x^2 + b\, y^2 + c\, z^2 + d\, w^2$$

$(i.e. 54\ sets\ of\ values\ of\ \{a, b, c, d\})$ that are universal. In that research paper, Ramanujan discussed a ternary quadratic form of the type

$$x^2 + 10\, y^2 + z^2 = m.\ where\ x, y, z, m\ are\ integers$$

and provided interesting results on representability of even, odd integers. Several researchers such as Ken ono provided new results on such quadratic form. The author became interested in ternary quadratic form of the type

$$a^2 + 2\, b^2 + c^2$$

associated with a 2 x 2 symmetric matrix with integer elements i.e.

$$\bar{X} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

It readily follows that

$$Tr(\bar{X}^2) = a^2 + 2\, b^2 + c^2 = \mu_1^2 + \mu_2^2\ , where$$

$\mu_1, \mu_2$ are the real valued eigenvalues of 2 x 2 symmetric matrix $\bar{X}$. He derived interesting results on such a specific ternary quadratic form. On checking the literature of integral quadratic forms, the author realized that the ideas in [3,4] can readily be generalized to arbitrary dimension matrices. The culmination of such efforts is the current research paper.

This research paper is organized as follows. In Section 2, generalized Ramanujan quadratic forms are investigsted for interesting results. The conclusions are documented in Section 3.

## 2. "Canonical"/"Interesting" Ternary Quadratic Forms: Linear Algebraic Approach:

Consider an arbitrary ternary quadratic form specified as

$\alpha a^2 + \beta\, b^2 + \gamma\, c^2$, where $\alpha, \beta, \gamma$ are known constants and { a, b, c } are integer unknowns defining the quadratic form. It readily follows that with

$$\bar{X} = \begin{bmatrix} \sqrt{\alpha}\, a & \sqrt{\frac{\beta}{2}}\, b \\ \sqrt{\frac{\beta}{2}}\, b & \sqrt{\gamma}\, c \end{bmatrix},\ we\ have\ that\ Tr(\bar{X}^2) = \alpha\, a^2 + \beta\, b^2 + \gamma\, c^2 = \mu_1^2 + \mu_2^2\ , where$$

$\mu_1, \mu_2$ are the real valued eigenvalues of 2 x 2 symmetric matrix $\bar{X}$.

Note: It is possible to express $Tr(\bar{X}^2) = \alpha\, a^2 + \beta\, b^2 + \gamma\, c^2 = f_2(a, b, c\, ;\, \alpha, \beta, \gamma)$ for various possible 2 x 2 symmetric matrices $\bar{X}'$ s by a permutation of the constants $\alpha, \beta, \gamma$. Detailed enumeration of all such possible 2 x 2 matrices is avoided for brevity. It should be noted that such 2 x 2 symmetric matrices will have different eigenvalues $\delta_1, \delta_2$ such that

$$Tr(\bar{X}^2) = \alpha\, a^2 + \beta\, b^2 + \gamma\, c^2 = \delta_1^2 + \delta_2^2.$$

Now, we specifically consider the 2 x 2 symmetric matrix

$$\bar{X} = \begin{bmatrix} \sqrt{\alpha}\, a & \sqrt{\frac{\beta}{2}}\, b \\ \sqrt{\frac{\beta}{2}}\, b & \sqrt{\gamma}\, c \end{bmatrix}$$

and determine the eigenvalues of such a matrix in closed algebraic form. It readily follows that

$$\mu_1, \mu_2 = \frac{\sqrt{\alpha}\, a + \sqrt{\gamma}\, c \pm \sqrt{D}}{2}, \ where\ D = (\sqrt{\alpha}\, a - \sqrt{\gamma}\, c)^2 \pm 2\, \beta\, b^2 .$$

It should be noted that for the most general ternary quadratic form, the eigenvalues are determined using the above expression.

- **"Canonical"/"Interesting" Ternary Quadratic Forms:**

From the expression for eigenvalues, it is clear that if the eigenvalues must be integers or atmost "complimentary" quadratic surds, $\{ \alpha, \gamma \}$ are

(i) Perfect squares of integers: In this case, without loss of generality

$$Tr(\bar{X}^2) = a^2 + \beta\, b^2 + c^2$$ or

(ii) $\alpha = 1, \gamma = 1$.

Also, if $\beta$ is not a perfect square of integer, it can be

(I) An even number. In such case $\beta$ can be considered to be 2 r with r being an odd number which is not a perfect square of integer ( with simple reasoning, it is possible to absorb the square part of $\beta$ into b ). Thus, in this case, we have the canonical form

$$Tr(\bar{X}^2) = a^2 + 2\, r\, b^2 + c^2$$ with r being odd nimber .

$We\ call\ it\ TYPE-1\ CANONICAL\ TERNARY\ QUADRATIC\ FORM.$

(II) Or an odd number which is not a perfect square. As in (I) above, without loss of generality, we have the canonical form

$$Tr(\bar{X}^2) = a^2 + \tilde{r}\, b^2 + c^2\ with\ \tilde{r}\ being\ odd\ number.$$

$We\ call\ it\ TYPE-2\ CANONICAL\ TERNARY\ QUADRATIC\ FORM$

Now, we realize that Ramanujan ternary quadratic form is a special case ternary quadratic form with $\alpha = 1,\ \beta = 10, \gamma = 1$ .

In the spirit of Ramanujan ternary quadratic form, we propose the following:

Type 1 Generalized Ramanujan Ternary Quadratic form:

$$f_2(a, b, c\, ;\ p) = a^2 + 2\, p\, b^2 + c^2,\ where\ p\ is\ an\ odd\ prime.$$

Type 2 Generalized Ramanujan Ternary Quadratic form:

$$f_2(a, b, c\, ;\ p) = a^2 + 2\, r\, b^2 + c^2,\ where\ r\ is\ an\ odd\ number$$

$which\ is\ not\ square\ of\ an\ integer\,.$

Note: $r = 1, r = 2$ $are\ special\ cases$ i.e. r = 1 corresponds to a 2 x 2 symmetric matrix and r = 2 implies $f_2(a,b,c) = a^2 + (2\,b)^2 + c^2$, $a$ type of quadratic form studied by Lagrange.

We now arrive at novel results associated with type 1, type 2 Ramanujan Ternary quadratic forms associated with the 2 x 2 matrix

$$\bar{X} = \begin{bmatrix} a & \sqrt{r}b \\ \sqrt{r}\,b & c \end{bmatrix}.$$

In this case, the eigenvalues are given by the following closed form formula:

$$\mu_1, \mu_2 = \frac{a + c \pm \sqrt{D}}{2}\ ,\ where\ D = (a - c)^2 \pm 4\ r\,b^2\,.$$

Note: It is evident that D is the DISCRIMINANT of the quadratic characteristic Polynomial of the 2 x 2 symmetric matrix $\bar{X}$.

Now, we realize that r = 1 is a special case considered and studied in [ ]. Also, r=2 case provides new insights into ternary quadratic form considered by Lagrange. We first derive results associated with the case, when

(i) r is a prime
(ii) r is an odd number which is not perfect square of an integer.

We have the following definition:

Definition: A "canonical/interesting" Terrnary integral quadratic form is called DOUBLY INTEGRAL if the eigenvalues of associated 2 x 2 symmetric matrix $\bar{X}$ are integers.

We now determine the conditions for the eigenvalues of $\bar{X}$ to be integers. We consider the boundary cases, where a =1 or c= 1 or a = c = 1 later. We first consider the non-boundary cases

CASE 1: Suppose { a , c } are of opposite polarity i.e. a is even and c is odd or a is odd and c is even. From the expression for the eigenvalues $\mu_1, \mu_2$, it is clear that the discriminant D which is not a multiple of 4 is not a perfect square of integer ( since (a-c) is odd and $4\,r\,b^2$ is a multiple of 4 ). Hence, the eigenvalues are "complimenatry quadratic surds" and not integers.

CASE 2: Suppose { a , c } are of same polarity i.e. a is even and c is even or a is odd and c is odd . Further, let the discriminant D be a perfect square of integer. In such cases, it readily follows that the eigenvalues are both integers and their polarity must be same i.e.both eigenvalues are even or both eigenvalues are odd numbers.

CLAIM: Thus, leaving out the boundary cases, a necessary condition for the eigenvalues $\mu_1, \mu_2$ to be integers is that { a, c } must be of same polarity. Further the discriminant $D = (a - c)^2 \pm 4\ r\,b^2$ must be a perfect square of integer.

CLAIM: The eigenvalues of Generatlized Ramanujan Ternary Quadratic Form (GRTQF) are never RATIONAL NUMBERS.

This claim follows from the fact that when { a, c } are both of same polarity ( necessary condition for the eigenvalues to be integers ), and the eigenvalues are not complimentary quadratic surdes

$$\frac{(a+c)}{2} \text{ is an integer and } \frac{\sqrt{D}}{2} \text{ is also an integer.}$$

CLAIM: If the eigenvalues are integers ( i.e. { a, c} are of same polarity ( and hence $Tr(\bar{X}^2) = \mu_1^2 + \mu_2^2$ is an even number ), the eigenvalues ( none of which is equal to one ) must both be odd numbers or both must be even numbers . Futhermore, if one of the eigenvalues is equal to one, the other one must be an odd number.

Note: Suppose $a = c$ and r is an odd number which is not perfect square of an integer. It readily follows that the eigenvalues are “complimentary quadratic surds”.

In view of the above discussion, we state and prove the following Theorem:

**Theorem 1:** Consider a “canonical”/ interesting Ternary Quadratic Form of the type $f_2(a,b,c;r) = a^2 + 2\,r\,b^2 + c^2$, *where r is an odd number and* $a \neq 0, b \neq 0, c \neq 0$.
Let the associated 2 x 2 matrix be $\bar{X} = \begin{bmatrix} a & \sqrt{r}b \\ \sqrt{r}\,b & c \end{bmatrix}$ such that

$Tr(\bar{X}^2) = \mu_1^2 + \mu_2^2 = f_2(a,b,c;r)$, *where* $\mu_1, \mu_2$ *are the eigenvalues of* $\bar{X}$.
A. Let $\bar{X}$ be a non-singular matrix with integer eigenvalues and let

$f_m(a,b,c;r) = Tr(\bar{X}^m) = \mu_1^m + \mu_2^m$, be a tri-variate polynomial in { a,b, c } given an odd integer ‘r’. Under these conditions, the following claims hold true:

$(i)\ Tr(\bar{X}^m) = \mu_1^m + \mu_2^m \neq \mu_3^m$, for any integer $m \geq 2$. Thus, the associated Diophantine equations have no solutions in integers a,b,c ( given fixed r )

$(ii)\ Det\,(\bar{X}^m) = \mu_1^m \mu_2^m = (\mu_1\,\mu_2)^m = (Det(\bar{X}))^m = (ac - r\,b^2)^m$ *for all m*

B. If $\bar{X}$ *is singular matrix , then* $Tr(\bar{X}^m) = \delta^m$ *for all m* ( *where* $\delta = a + c$, *the non – zero eigenvalue of* $\bar{X}$ ).

Proof: We prove the theorem in various cases. We first consider $r \geq 3$ *case*.

- Suppose, one of the eigenvalues is equal to 1. Since $\mathrm{Tr}(\bar{X}) = a + c$ *is an integer*, for the discriminant to be square of an integer, { a, c } must both be odd or both must be even numbers Hene the other eigenvalue must be an odd number. In such case, it readily follows that
$$Tr(\bar{X}^2) = \mu_1^2 + \mu_2^2 = 1 + (2\,l + 1)^2 \text{ for an integer } 'l'. \text{ Thus, we have that } Tr(\bar{X}^2) = 4\,(l^2 + l) + 2 \text{ . Hence, } Tr(\bar{X}^2) \text{ cannot be square of an even or odd integer.}$$
- Now, consider the case where none of the eigenvalues is equal to 1. Since, we are considering the case where the eigenvalues are integers, { a, c } must both be even or odd. In such case, both the eigenvalues must be even or both or odd. Hence, we now reason that in such case, ‘b’ must be even ( with { a, c } of same polarity : even/odd ). We readily have that

$$\mu_1, \mu_2 = \frac{a + c \pm \sqrt{D}}{2}, \quad where\ D = (a - c)^2 \pm 4\, r\, b^2.$$

For convenience in discussion, we define the following variables

$$e = \frac{a+c}{2}, \quad f = \frac{a-c}{2} \quad and \quad \widetilde{D} = f^2 + r\, b^2$$

Consider the case where { a, c } are both even i.e. the eigenvalues $\mu_1, \mu_2$ are both even or both odd numbers. In this case, with 'r' being an odd number, we can have $e = \frac{a+c}{2}$ to be an even number or an odd number.

( We consider the cases where b=1, $\frac{a+c}{2}$ or $\frac{a-c}{2}$ equal 1 separately in the following discussion )

(i) Let $e = \frac{a+c}{2}$ be an even number. For $\widetilde{D}$ to be an even number ( to ensure that $\mu_1, \mu_2\ are\ even$ ), we necessarily require that 'b' is an even number ( since $f = \frac{a-c}{2}$ is an even number other than being 1 and r is an odd number.

(ii) Now, suppose $e = \frac{a+c}{2}$ is an odd number ( Hence, $f = \frac{a-c}{2}$ is an odd number other than being 1 ). . For $\widetilde{D}$ to be an even number ( to ensure that $\mu_1, \mu_2\ are\ even$ ), we necessarily require that 'b' is an even number.

Even in the case, where { a, c } are both odd, based on same reasoning as above ( for both eigenvalues to be even or odd ), we reason that 'b' must be an even number

From the above two cases, we must have that { a, c } are both even or both odd and 'b' must be an even number ( b=1 case is considered separately. We now reason, in the following discussion that in such cases, $Tr(\bar{X}^2)$ is not perfect square of an integer.

Consider the case where { a, c } are both odd and b is even. In this case, we have that r

$$Tr(\bar{X}^2) = a^2 + 2\, r\, b^2 + c^2 = (2l + 1)^2 + 2\,(2t + 1)\,(2n)^2 + (2m+1)^2$$

$$with\ l, m, n, t\ being\ integers.\ On\ simplification, \quad it\ readily\ follows\ that$$
$$Tr(\bar{X}^2) = a^2 + 2\, r\, b^2 + c^2 = 4\,T + 2, \quad where\ T\ is\ an\ integer.\ Hence, Tr(\bar{X}^2)$$
$$cannot\ be\ square\ of\ an\ even\ or\ odd\ integer.$$

Note: The case considered here doesnot assume that the eigenvalues are integers. Hence, even if the eigenvalues are "complimentary" quadratic surds, in this case $Tr(\bar{X}^2)$ is not a perfect square of integer.

We are now left with the case, where { a, c } are both even and b is even. r is odd. Let $a = 2l,\ c = 2m,\ b = 2n,\ r = 2\,t + 1$. Hence,

$$Tr(\bar{X}^2) = a^2 + 2\, r\, b^2 + c^2 = (2l)^2 + 2\,(2t + 1)\,(2n)^2 + (2m)^2$$
$$= 4\,(l^2 + 2\, r\, n^2 + m^2).$$

If $l, m$ *are odd and* $n$ *is even*, the above discussed case ( based argument ) gives the desired result that RHS is not a perfect square of integer. Based on above discussion, we reason that
$a^2 + 2\,r\,b^2 + c^2$ can always be expressed as
$(\textit{square of an even number})(\tilde{l}^2 + 2\,r\,\tilde{n}^2 + \tilde{m}^2)$ where the integers, $\{\tilde{l}, \tilde{m}, \tilde{n}\}$ are the elements of 2 x 2 symmetric matrix
$\tilde{X} = \begin{bmatrix} \tilde{l} & \tilde{n}\sqrt{r} \\ \tilde{n}\sqrt{r} & \tilde{m} \end{bmatrix}$ . With such a matrix, the case where $\tilde{l}$ *or* $\tilde{m}$ are 1 can be ruled out by the above discussion. Also, we are effectively led to the case where $\tilde{l}$ *is odd*, $\tilde{m}$ is odd and $\tilde{n}$ is even. Hence, $Tr(\bar{X}^2)$ will not be square of an integer. This argument resembles method of infinite descent of Fermat. The reasoning is mainly dependent on the fact that when the eigenvalues are both integers,
$Tr(\bar{X}^2) = e^2 + f^2 + r\,b^2$ , *where* $e = \frac{a+c}{2}$ $, f = \frac{a-c}{2}$.
Note: Suppose r=1 and b=1. We consider the case where { a, c } are both even or both odd ( i.e. a+c, a-c are both even numbers. Hence the eigenvalues must both be even or both odd ). For convenience, let

$$e = \frac{a+c}{2}\,, \qquad f = \frac{a-c}{2} \quad \textit{and} \;\; \widetilde{D} = f^2 + 1.$$

Clearly, if f=1, the discriminant D =2 is not square of integer. Hence, the eigenvalues cannot be integers. Also, if e is even, D is odd and the eigenvalues cannot be integers. Similarly, if e is odd, D is odd and eigenvalues cannot be integers.

Thus, in this case the case the conditions of the above Theorem are violated.
Note: In the boundary case, a=1 or c= 1, similar reasoning as above leads to the conclusion that $Tr(\bar{X}^2)$ *is not perfect square of an integer*. Details are avoided for brevity.

Corollary: (i) Suppose the eigenvalues $\mu_1, \mu_2$ are integers and $m \geq 3,\ l \geq 2$ . From similar argument ( as above ), it is possible to reason that $\mu_1^m + \mu_2^m \neq \mu_3^l$

- THEOREM FOR CANONICAL TERNARY QUADRATIC FORMS OF TYPE 2:

  In the spirit of Theorem 1, Theorem 2 readily follows for the second type of "canonical"/interesting Ternary quadratic forms i.e.

$$Tr(\bar{X}^2) = f_2(a, b, c; r) = a^2 + r\,b^2 + c^2,$$

*The theorem statement follows verbatim with regard to representation of an integer by* $Tr(\bar{X}^2)$ *and higher degree forms. Detailed statement of Theorem is avoided for brevity.*

- **GENERALIZATION TO HIGHER DIMENSIONAL SYMMETRIC SQUARE MATRICES: Higher order quadratic and higher degree forms: Generalized Waring Problem in eigenvalues:**

From the above discussion for 2 x 2 square matrices, it readily follows that a quaternary quadratic form can be associated with an interesting 3 x 3 square matrix. Even in this case, we run into the case where the eigenvalues are integers ( which can be expressed by a closed form algebraic formula ). In the spirit of Theorem 1, Theorem 2 new theorem could be proved. When we consider an M x M symmetric matrix, we can represent an $L^{th}$ *order i.e* $L = \frac{M(M+1)}{2}$ quadratic/ Higher degree form using M eigenvalues only i.e.

$$Tr(X^m) = f_m(a, b, c, \dots, \qquad) = \sum_{i=1}^{M} \mu_i^m$$

- In the spirit of EXISTING THEMES IN INTEGER QUADRATIC FORMS SUCH AS UNIVERSALITY OF A QUADRATIC FORM ( e.g. 15, 290 theorems ), detailed results are being documented in [4]. Several new ideas explored in [5] are being generalized

## 3. CONCLUSIONS:

In this research paper, the author identified two canonical ternary quadratic forms in the spirit of Ramanujan Ternary quadratic form. Interesting Theorems related to representation by such forms are proved using the associated 2 x 2 symmetric matrix. The results are being generalized to quadratic/higher degree forms associated with higher dimensional symmetric matrices

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