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Abstract

Grönwall's function G is defined for all natural numbers $n > 1$ by $G(n) = \frac{\sigma(n)}{n \cdot \log \log n}$ where $\sigma(n)$ is the sum of the divisors of n and \log is the natural logarithm. We require the properties of extremely abundant numbers, that is to say left to right maxima of $n \mapsto G(n)$. We also use the colossally abundant and hyper abundant numbers. There are several statements equivalent to the famous Riemann hypothesis. We state that the Riemann hypothesis is true if and only if there exist infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. Using this new criterion, we prove that the Riemann hypothesis is true.

Keywords: Riemann hypothesis, Extremely abundant numbers, Colossally abundant numbers, Hyper abundant numbers, Arithmetic functions

MSC Classification: 11M26 , 11A25

1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d|n} d,$$

where $d \mid n$ means the integer d divides n . In 1997, Ramanujan's old notes were published where it was defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [1]. A natural number n is called superabundant precisely when, for all natural

A Millennium Prize Problem

2 The Riemann hypothesis

numbers $m < n$

$$\frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}.$$

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \text{ for } (m > 1).$$

Every colossally abundant number is superabundant [2]. Let us call hyper abundant an integer n for which there exists $u > 0$ such that

$$\frac{\sigma(n)}{n \cdot (\log n)^u} \geq \frac{\sigma(m)}{m \cdot (\log m)^u} \text{ for } (m > 1),$$

where \log is the natural logarithm. Every hyper abundant number is colossally abundant [3, pp. 255]. In 1913, Grönwall studied the function $G(n) = \frac{\sigma(n)}{n \cdot \log \log n}$ for all natural numbers $n > 1$, [4]. We have the Grönwall's Theorem:

Proposition 1

$$\limsup_{n \rightarrow \infty} G(n) = e^\gamma$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [4].

Next, we have the following Robin's results:

Proposition 2 Let $3 \leq N < N'$ be two consecutive colossally abundant numbers, then

$$G(n) \leq \text{Max}(G(N), G(N'))$$

when satisfying $N < n < N'$ [5, Proposition 1 pp. 192].

Proposition 3 There are infinitely many colossally abundant numbers N such that $G(N) > e^\gamma$ when the Riemann hypothesis is false [5, Proposition 1 pp. 204]. There exist infinitely many colossally abundant numbers N such that $G(N) < e^\gamma$ [5, Theorem 1 pp. 188], [5, Proposition 1 pp. 204].

There are champion numbers (i.e. left to right maxima) of the function $n \mapsto G(n)$:

$$G(m) < G(n)$$

for all natural numbers $10080 \leq m < n$. A positive integer n is extremely abundant if either $n = 10080$, or $n > 10080$ is a champion number of the function $n \mapsto G(n)$. In 1859, Bernhard Riemann proposed his hypothesis [6]. Several analogues of the Riemann hypothesis have already been proved [6].

Proposition 4 The Riemann hypothesis is true if and only if there exist infinitely many extremely abundant numbers [7, Theorem 7 pp. 6].

We use the following property for the extremely abundant numbers:

Proposition 5 *Let $N < N'$ be two consecutive colossally abundant numbers and $n > 10080$ is some extremely abundant number, then N' is also extremely abundant when satisfying $N < n < N'$ [7, Lemma 21 pp. 12].*

This is our main theorem

Theorem 1 *The Riemann hypothesis is true if and only if there exist infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$.*

Putting all together yields a new criterion for the Riemann hypothesis. Now, we can conclude with the following result:

Theorem 2 *The Riemann hypothesis is true.*

Proof We consider two large enough consecutive colossally abundant numbers $N < N'$. Let's assume that N' is a hyper abundant number with a parameter $u > 0$. This is possible since every hyper abundant number is colossally abundant [3, pp. 255]. In this way, we obtain that

$$\left(\frac{\log N}{\log N'}\right)^u \geq \frac{\left(\frac{\sigma(N)}{N}\right)}{\left(\frac{\sigma(N')}{N'}\right)}$$

by definition of hyper abundant numbers. That is

$$u \geq \frac{\log\left(\frac{\left(\frac{\sigma(N)}{N}\right)}{\left(\frac{\sigma(N')}{N'}\right)}\right)}{\log\left(\frac{\log N}{\log N'}\right)}$$

which is

$$1 \geq \frac{\log\left(\frac{\left(\frac{\sigma(N)}{N}\right)}{\left(\frac{\sigma(N')}{N'}\right)}\right)}{\log\left(\frac{\log N}{\log N'}\right)}$$

since $1 \geq u$ for large enough hyper abundant number N' by Lemma 2. That is

$$\frac{\log N}{\log N'} \geq \frac{\left(\frac{\sigma(N)}{N}\right)}{\left(\frac{\sigma(N')}{N'}\right)}.$$

However, we know that

$$\frac{\log \log N}{\log \log N'} > \frac{\log N}{\log N'}$$

4 *The Riemann hypothesis*

since $e < \log N < \log N'$ by Lemma 1. Finally, we obtain that

$$\frac{\log \log N}{\log \log N'} > \frac{\left(\frac{\sigma(N)}{N}\right)}{\left(\frac{\sigma(N')}{N'}\right)}.$$

which means $G(N) < G(N')$. Note also that, for all $u > 0$ [3, pp. 254]:

$$\lim_{n \rightarrow \infty} \frac{\sigma(n)}{n \cdot (\log n)^u} = 0$$

and so, there are infinitely many hyper abundant numbers. In this way, there are infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. Finally, the proof is complete by Theorem 1. \square

2 Central Lemma

Lemma 1 For two real numbers $y > x > e$:

$$\frac{y}{x} > \frac{\log y}{\log x}.$$

Proof We have $y = x + \varepsilon$ for $\varepsilon > 0$. We obtain that

$$\begin{aligned} \frac{\log y}{\log x} &= \frac{\log(x + \varepsilon)}{\log x} \\ &= \frac{\log\left(x \cdot \left(1 + \frac{\varepsilon}{x}\right)\right)}{\log x} \\ &= \frac{\log x + \log\left(1 + \frac{\varepsilon}{x}\right)}{\log x} \\ &= 1 + \frac{\log\left(1 + \frac{\varepsilon}{x}\right)}{\log x} \end{aligned}$$

and

$$\begin{aligned} \frac{y}{x} &= \frac{x + \varepsilon}{x} \\ &= 1 + \frac{\varepsilon}{x}. \end{aligned}$$

We need to show that

$$\left(1 + \frac{\log\left(1 + \frac{\varepsilon}{x}\right)}{\log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

which is equivalent to

$$\left(1 + \frac{\varepsilon}{x \cdot \log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

using the well-known inequality $\log(1 + x) \leq x$ for $x > 0$. For $x > e$, we have

$$\frac{\varepsilon}{x} > \frac{\varepsilon}{x \cdot \log x}.$$

In conclusion, the inequality

$$\frac{y}{x} > \frac{\log y}{\log x}$$

holds on condition that $y > x > e$. \square

3 Main Insight

Lemma 2 Every large enough hyper abundant number n is defined over a parameter $1 \geq u > 0$.

Proof Every large enough hyper abundant number n is defined over a parameter $u > 0$ as follows,

$$\frac{\sigma(n)}{n \cdot (\log n)^u} \geq \frac{\sigma(m)}{m \cdot (\log m)^u} \text{ for } (m > 1).$$

Then, we would have

$$\frac{\sigma(n)}{n \cdot (\log n)^u} \geq \frac{\sigma(n')}{n' \cdot (\log n')^u} \text{ for } (n > n' > e).$$

Thus,

$$\left(\frac{\log n'}{\log n} \right)^u \geq \frac{\frac{\sigma(n')}{n'}}{\frac{\sigma(n)}{n}}.$$

By Proposition 1, we know there exists some $n' \leq 5040$ such that

$$\frac{\frac{\sigma(n')}{n'}}{\frac{\sigma(n)}{n}} \geq \frac{\log \log n'}{\log \log n}$$

for large enough hyper abundant number n . Hence, we would obtain that

$$\left(\frac{\log n'}{\log n} \right)^u \geq \frac{\log \log n'}{\log \log n}.$$

Moreover, we know that

$$\left(\frac{\log \log n'}{\log \log n} \right)^u > \left(\frac{\log n'}{\log n} \right)^u$$

since $n > n' > e$ by Lemma 1. However, this implies that

$$\left(\frac{\log \log n'}{\log \log n} \right)^u > \frac{\log \log n'}{\log \log n}.$$

which immediately forces the parameter u to be necessarily lesser than or equal to 1. □

4 Proof of Theorem 1

Proof Suppose there are not infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. This implies that the inequality $G(N) \geq G(N')$ always holds for a sufficiently large N when $N < N'$ is a pair of consecutive colossally abundant numbers. That would mean the existence of a single colossally abundant number $N'' \geq 10080$ such that $G(n) \leq G(N'')$ for all natural numbers $n > N''$ according to Proposition 2. Certainly, the existence of such single colossally abundant number N'' is because of the Grönwall's function G would become decreasing on colossally abundant numbers starting from some single value. We use the Proposition 5 to reveal that under these preconditions, then there are not infinitely many extremely abundant numbers. This implies that the Riemann

6 *The Riemann hypothesis*

hypothesis is false as a consequence of Proposition 4. By contraposition, if the Riemann hypothesis is true, then there exist infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$.

Suppose that there exist infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. On the one hand, let's assume from these infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$, then there could be only a finite amount of these N' such that $e^\gamma < G(N')$. Thus, we deduce there could be only a finite amount of colossally abundant numbers N'' such that $e^\gamma < G(N'')$. However, when the Riemann hypothesis is false, then there are infinitely many colossally abundant numbers N'' such that $e^\gamma < G(N'')$ by Proposition 3. On the other hand, let's assume from these infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$, then there could be an infinite amount of these N' such that $e^\gamma < G(N')$.

Based on this opposite assumption, it could appear the possible scenarios:

- there would be an infinite increasing subsequence of colossally abundant numbers N_i such that $e^\gamma < G(N_i)$ and $G(N_i) < G(N_{i+1})$,
- or there would be a colossally abundant number N'' such that for all colossally abundant numbers $N > N''$ we have $e^\gamma \leq G(N)$,
- or there would be infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < e^\gamma < G(N')$.

However, it cannot exist an infinite increasing subsequence of colossally abundant numbers N_i such that $e^\gamma < G(N_i)$ and $G(N_i) < G(N_{i+1})$, by Proposition 1 and the properties of limit superior. Moreover, there cannot be a colossally abundant number N'' such that for all colossally abundant numbers $N > N''$ we have $e^\gamma \leq G(N)$, since this implies that there are not infinitely many colossally abundant numbers N''' such that $G(N''') < e^\gamma$ which is a contradiction by Proposition 3.

Furthermore, there are not infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < e^\gamma < G(N')$. Certainly, we deduce that

$$\frac{G(N')}{G(N)} = 1 + \frac{G(N') - G(N)}{G(N)} > 1 + \frac{G(N') - G(N)}{e^\gamma} > 2 - \frac{G(N)}{e^\gamma}.$$

Under our assumption that $G(N) < e^\gamma < G(N')$ we obtain a contradiction since the expression $\frac{G(N')}{G(N)}$ tends to be smaller than or equal to $2 - \frac{G(N)}{e^\gamma}$ when $G(N) < e^\gamma$ and as N grows to infinity by Proposition 1. Therefore, the Riemann hypothesis would be true when there exist infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. \square

5 Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. Undoubtedly, this is a breakthrough result.

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